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FOREWORD

This is the third of three volumes of the final report prepared by the Research Triangle Institute, Durham, North Carolina under NASA contract NASw-905, "Development of Reliability Methodology for Systems Engineering". This work was administered under the technical direction of the Office of Reliability and Quality Assurance, NASA Headquarters with Mr. John E. Condon, Director, as technical contract monitor.

The work described in this report was conducted by M. R. Leadbetter and J. D. Cryer. The emphasis of this work has been to develop mathematical methods for the analysis of stochastic data, from a reliability standpoint. The methodology thus developed extends the theory available from work done under a previous NASA contract (NASw-334). While the results obtained are theoretical in nature, the requirements for practical application have been kept constantly in mind. In particular, simple asymptotic approximations have been given for certain important results which would otherwise lead to difficult computational problems.

The contents of this report have also been submitted as Mr. Cryer's dissertation in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Statistics at the University of North Carolina.

PREFACE

The objective of this contract was to develop reliability methodology which relates to various techniques which can be applied in designing reliabile systems and to extend the methodology by the development and demonstration of new techniques. It was important to have available a system on which to test and demonstrate the results. A complex static inverter was chosen for this purpose and served this role well.

The three major areas of effort in the program are defined by the titles of the final report volumes listed as follows:

Vol. I. Methodology: Analysis Techniques and Procedures

Vol. II. Application: Design Reliability Analysis of a 250 Volt-Ampere Static Inverter

Vol. III. Theoretical Investigations: An Approach to a Class of Reliability Problems

The purpose of Vol. I is to describe the mathematical techniques which are available for performing the reliability analysis of equipment life and performance. Appropriate technique selection, coupled with proper coordination of efforts during design, are essential for engineering reliability into equipment. Vol. II considers the practical application of reliability analysis to circuit design and demonstrates improvements in the identification and solution of problems using the techniques described in Vol. I. This employs the static inverter as an example. Vol. III describes fundamental studies in stochastic processes related to reliability.

Other technical reports issued under this contract effort are as follows:

1. "On Certain Functionals of Normal Processes," Technical Report No. 1, September 1964.
2. "Functional Description of a 250 Volt-Ampere Static Inverter," Technical Report No. 2, December 1964.
3. "The Variance of the Number of Zeros of Stationary Normal Processes," Technical Report No. 3, March 1965.
4. "Problems in Probability," Technical Report No. 4, October 1965.
5. "Reliability Analysis of Timing Channel Circuits in a Static Inverter," Technical Report No. 5, December 1965.
6. "Reliability Analysis of Timing Channel Circuits in a Static Inverter," Technical Report No. 6, January 1966.

ABSTRACT

This report is concerned with certain random quantities derived from a continuous time stochastic process X(t). Particular interest is centered on the number of crossings of certain barriers by the sample functions of X(t) and other closely ralated random variables. Such quantities are of interest in reliability theory both in their own right as performance measures and because they can provide bounds on certain probabilities.

The basic definitions and fundamental relations for crossings are given first. It is noted that the actual distribution of the number of crossings is obtainable in only very special cases. In particular, this distribution is derived when the process is the so-called random cosine wave. In view of the difficulty in deriving the distributions involved, certain moments are considered. Assuming that X(t) is a non-stationary normal process, a formula for the mean number of crossings of a (continuously differentiable) curve is obtained under essentially minimal conditions. An expression for the second moment of the number of upcrossings of a level by a stationary normal process is derived next. It is shown by an example that the sufficient conditions given for the finiteness of this moment are also very close to being necessary. Incidental results in this connection include a formula for the covariance between the number of upcrossings in one interval and the number in another interval.

Also considered are certain random variables related to excursions outside of given barriers. The first two moments of these variables are given and it is noted that Chebyshev-type bounds on probabilities of interest can be obtained from these moments. Some discussion is devoted to crossings and excursions outside two-sided barriers.

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CHAPTER I

INTRODUCTION

We shall be concerned throughout with a stochastic process $X(t)$ having a continuous time parameter t. In particular we are interested in the number of crossings of certain barriers by the sample functions of $X(t)$ and in other closely related random variables. The interest in crossing problems came, initially, from electrical engineering applications, especially in radio propagation and communications theory. For example, the statistical properties of the number of zero crossings per unit time of a voltage waveform provides knowledge necessary for the design of certain FM signal detectors. Studies of fatigue in structures, analysis of flight test data and guidance systems, and investigations into speech waveforms provide further applications of interest.

Certain aspects of reliability theory may also be approached through crossing problems. When one is concerned with the quality of performance of a complex physical mechanism subject to random disturbing influences, it is sometimes convenient to consider "performance measures" which are based on the characteristics of a stochastic process associated with the system. For example suppose $X(t)$ denotes an output of the system and for good performance it is desirable that $X(t)$ remain below the level a during the operational period of the system, $t \in (0,t_o)$. Then we would like to evaluate $P\{X(t) < a,\ t \in (0,t_o)\}$. In terms of crossings this is the same as $P\{X(0) < a$ and $X(t)$ has <u>no crossings</u> of a for $t \in (0,t_o)\}$. The mean number of crossings of the level a during $(0,t_o)$ would also be of interest as a performance measure; the smaller the mean the better the performance from this point of view.

It is also easy to see that crossings are closely related to certain first-passage problems. Let T_a denote the first time after $t = 0$ at which $X(t) = a$. For processes with continuous sample functions T_a is a random variable and a first-passage problem consists of finding the distribution of T_a. We have the relation

$$P\{T_a > \tau\} = P\{\text{no crossing of } a \text{ for } t \in (0,\tau)\} .$$

First-passage problems are extremely difficult in general and even for stationary normal processes have only been solved for a few special cases. (See Slepian (1962) and Mehr and McFadden (1965) and the references contained therein.) We may however note also that

$$P\{T_a > \tau\} = P\{\max_{(0,\tau)} X(t) < a\} + P\{\min_{(0,\tau)} X(t) > a\} .$$

The asymptotic distribution of $\max_{(0,\tau)} X(t)$ as $\tau \to \infty$ has recently been obtained by Cramér (1965, 1966) for a stationary normal process under weak conditions.

Similarly in almost all cases the distribution of the number of crossings is not known. Thus our results will mainly be concerned with moments. However the moments are useful in their own right and indeed can be used to provide bounds and approximations to the probabilities discussed above.

In Chapter II the basic definitions are given and some of the fundamental relations derived. An approximation to a process $X(t)$ is described and shown to be useful in obtaining results concerning crossings. In the last section of Chapter II the actual distribution of the number of upcrossings is derived for the very special case of a "random cosine wave."

Chapter III is concerned with the mean number of curve crossings by a general non-stationary normal process. A formula for the mean is given and corresponding results for upcrossings and downcrossings are noted. Conditions sufficient for the mean number of crossings to be infinite are obtained and it is shown that the conditions under which the formula for the mean is obtained are close to being necessary. Some examples are given.

The second moment of the number of upcrossings of a level by a stationary normal process is found in Chapter IV. Conditions are given such that this moment is finite and it is shown by example that a very slight relaxation of these conditions leads to an infinite second moment. Hence again the sufficient conditions for finiteness are almost the necessary conditions. In 4.3 the variance of the total number of crossings is obtained from the corresponding result for upcrossings. Next the covariance between the number of upcrossings in one interval and the number in another interval is found. The remaining two sections give the asymptotic form for the variance of the number of crossings in an interval of length T as $T \longrightarrow \infty$ and some numerical results which indicate that in many cases of practical interest the asymptotic form is approached rather rapidly.

Chapter V deals with certain random variabes related to excursions outside of a curve a(t). The mean and variance of the variables are derived and it is shown how they can give Chebyshev-type bounds on probabilities of interest. Some asymptotic formulae and numerical calculations end this chapter.

In Chapter VI we indicate how the results of the earlier chapters can be extended to two-sided barriers. That is, we consider the total number of crossings of the two levels a and b and excursions outside such boundaries.

Finally, in most cases the conditions assumed are sufficient to ensure the existence of a separable version of the process under consideration. However, in any case, separability will be assumed throughout without further comment.

CHAPTER II

FUNDAMENTALS

2.1 Crossings

In this and the following two chapters we will discuss certain properties of "crossings" of levels or curves by the sample functions of a stochastic process. Although the meanings of such works as "crossings" or "upcrossings" are intuitively clear they must of course be defined precisely within a mathematical framework. In particular it might seem reasonable to define an upcrossing in terms of the process value and the sign of its derivative. However, to obtain results under minimal conditions it is sometimes necessary to avoid assuming that the process has (with probability one) a sample derivative. Thus we make the following definitions which include those given by Volkonski (1960), Ylvisaker (1965) and Leadbetter (1966a).

Suppose $x(t)$ is a (non-random) continuous real function for $t \in [0,1]$ with the property that $x(k2^{-n}) \neq 0$ for $k = 0,1,\ldots,2^n, n = 1,2,\ldots$ Then $t_o \in [0,1]$ is said to be a <u>zero crossing</u> of $x(t)$ if for every $\epsilon > 0$ there are points t_1, t_2 in $[0,1]$ such that $t_o - \epsilon \leq t_1 \leq t_o \leq t_2 \leq t_o + \epsilon$ and $x(t_1)x(t_2) < 0$. More generally, if $a(t)$ is a continuous function on $[0,1]$ and $a(k2^{-n}) \neq x(k2^{-n})$, $x(t)$ is said to have a <u>crossing of $a(t)$</u> at t_o if $x(t)-a(t)$ has a zero crossing at t_o. Note that though we do not assume $x(t_o) = 0$ at a zero crossing t_o, nevertheless this follows from the continuity of the function $x(t)$. For if $x(t_o) \neq 0$ it follows that $x(t)$ is strictly positive or else strictly negative in some neighborhood of t_o and hence cannot have a zero crossing there.

Clearly as our definition stands it does not include points of tangency (i.e. points where $x(t_o) = 0$ that are not crossings) to the axis (or the curve a(t)) as zero crossings. However, the definition is the most convenient one and in many applications of crossings to random functions it can be shown that the set of sample functions which are somewhere tangential to the axis (or curve) has zero probability of occurring. Hence quantities such as moments of the number of crossings are also the moments of the number of times the function (process) actually assumes the value. More will be said of this as needed.

It will also be convenient to define two special kinds of crossings. The point t_o is called an <u>upcrossing</u> of $x(t)$ if there exists $\delta > 0$ such that $x(t) \leq 0$ when $t_o - \delta \leq t \leq t_o$ and $x(t) \geq 0$ when $t_o \leq t \leq t_o + \delta$. Similarly the function $x(t)$ has a <u>downcrossing</u> at t_o if there exists $\delta > 0$ such that $x(t) \geq 0$ for $t_o - \delta \leq t \leq t_o$ and $x(t) \leq 0$ for $t_o \leq t \leq t_o + \delta$.

The following rather intuitive lemmas will be needed later. Similar results are used (without full proof) by Leadbetter (1966a). (see also Ylvisaker (1965)).

<u>Lemma</u> 2.1.1: If $x(t_1).x(t_2) < 0$ then $x(t)$ has a zero crossing between t_1 and t_2.

Proof: Suppose this is <u>not</u> true; then every point of the interval $[0,1]$ lies in an open interval in which $x(t)$ has constant sign. The collection of all such intervals is an open covering of $[0,1]$. By the Heine-Borel theorem there exists a finite subcover. On overlapping intervals $x(t)$ must have the same sign and thus, by finite induction, $x(t_1)$ and $x(t_2)$ must have the same sign - a contradiction.

Let N, N_u, N_d denote, respectively, the number of crossings, upcrossings, and downcrossings of $x(t)$ for $t \in [0,1]$. In general there may be crossings which are neither upcrossings nor downcrossings - consider $x(t) = (t-t_o)\sin(1/(t-t_o))$ with $t_o \in [0,1]$ and not of the form $k2^{-n}$. If, however, N is finite we have the following result.

<u>Lemma</u> 2.1.2: If $N < \infty$ then $N = N_u + N_d$.

Proof: Let t_o be a zero crossing of $x(t)$. Since $N < \infty$ there is an interval $(t_o-\epsilon, t_o+\epsilon)$ which contains no other crossings. The function $x(t)$ cannot change sign in $(t_o-\epsilon, t_o)$ since if it did there would, by Lemma 2.1.1, be another crossing in $(t_o-\epsilon, t_o)$. Similarly $x(t)$ has constant sign on $(t_o, t_o+\epsilon)$ and thus t_o is either an upcrossing or a downcrossing.

Lemma 2.1.3: If $N < \infty$ and $x(t_1) < 0 < x(t_2)$ where $t_1 < t_2$, then $x(t)$ has an upcrossing between t_1 and t_2.

Proof: By Lemma 2.1.1 at least one crossing occurs between t_1 and t_2. Let t_o be the first such crossing. By Lemma 2.1.2 t_o is either an upcrossing or a downcrossing. Suppose it is a downcrossing. Then there is a point between t_1 and t_o where $x(t)$ is strictly positive and hence, by Lemma 2.1.1 there is another crossing before t_o - a contradiction. Thus t_o is an upcrossing.

Lemma 2.1.4: If $N < \infty$, then $N_u - N_d$ assumes only the values $0, \pm 1$.

$$N_u - N_d = \begin{cases} +1 & \text{iff } x(0) < 0 < x(1) \\ -1 & \text{iff } x(0) > 0 > x(1) \\ 0 & \text{otherwise} \end{cases}$$

Proof: Using Lemma 2.1.1 again we see that there must be a crossing between any two upcrossings (downcrossings). Thus the upcrossings and downcrossings alternate and the lemma easily follows from this fact.

2.2 The Piecewise Linear Approximation.

For any $t \in [0,1]$ and any positive integer n, we let $k = k_n(t)$ denote the unique integer such that $k2^{-n} \leq t < (k+1)\, 2^{-n}$. Then for each n we define $y_n(t)$, the piecewise linear approximation to $x(t)$, as the function

$$y_n(t) = x(k2^{-n}) + 2^n(t-k2^{-n})[x((k+1)2^{-n}) - x(k2^{-n})] , \tag{2.2.1}$$

that is, $y_n(t) = x(t)$ for t of the form $k2^{-n}$ ($k = 0,1,\ldots 2^n$) and $y_n(t)$ is linear between such points.

Let N_n denote the number of zero crossings by $y_n(t)$ for $t \in [0,1]$. Then the following useful lemmas can be obtained.

<u>Lemma</u> 2.2.1: For each n we have $N_n \leq N_{n+1} \leq N$.

Proof: If $y_n(t)$ has a zero crossing at t_o then there is an integer k such that $k2^{-n} < t_o < (k+1)2^{-n}$ and $x(k2^{-n})\cdot x((k+1)2^{-n}) < 0$. Thus by Lemma 2.1.1 $x(t)$ has a zero crossing in $(k2^{-n}, (k+1)2^{-n})$ and thus $N_n \leq N$. Further $x((2k+1)2^{-n-1})$ is either positive or negative so that $y_{n+1}(t)$ also has a zero crossing in $(k2^{-n}, (k+1)2^{-n})$, i.e. $N_n \leq N_{n+1}$.

<u>Lemma</u> 2.2.2: $N_n \uparrow N$ as $n \longrightarrow \infty$.

Proof: If N is finite let $t_1, t_2, \ldots, t_N$ be the zero crossings of $x(t)$. There is a set of N disjoint intervals each containing one of the t_i. Let $a < t_i < b$ be a typical such interval. By the definition of zero crossing there exist two points τ_1, τ_2 such that $a < \tau_1 < t_i < \tau_2 < b$ and $x(\tau_1)\cdot x(\tau_2) < 0$. Since $x(\cdot)$ is continuous there are neighborhoods (c,d), (e,f) of τ_1, τ_2 such that $x(\cdot)$ is of contstant sign on each one. Now the set of points $\{k2^{-n}: k=0,1,\ldots 2^n, n=1,2,\ldots\}$ is dense in [0,1] and thus there are integers n, k_1, k_2 with $0 \leq k_1, k_2 \leq 2^n$ such that $c < k_1 2^{-n} < d$ and $e < k_2 2^{-n} < f$. Therefore $y_n(k_1 2^{-n})\cdot y_n(k_2 2^{-n}) < 0$ and $y_n(t)$ has a zero crossing in (a,b) also. Repeating this argument for each t_i yields $\lim_{n \to \infty} N_n = N$.

Now suppose N is infinite. Let M be an arbitrary positve integer and let $t_1, t_2, \ldots, t_M$ be zero crossings of $x(t)$. The arguments in the first part of the proof show immediately that $N_n \geq M$ as soon as n is sufficiently large. Thus N_n tends to infinity as required.

Let us now consider crossings by a stochastic process $\{X(t): 0 \leq t \leq 1\}$. In order that crossings be defined we can here make the minimal assumptions that $X(t)$ has, with probability one, continuous sample functions and has continuous one-dimensional distributions. Under these assumptions we have that, with probability one, $X(t)$ is a continuous function and $X(k2^{-n}) \neq 0$ for $k = 0,1,\ldots 2^n$, $n = 1,2\ldots$ and therefore the previous lemmas will apply.

To show that N, the number of zero crossings by $X(t)$, is indeed a random variable we need to temporarily write $X(t)$ as $X(t,\omega)$ to explicitly show the dependance of $X(t)$ on the "sample point" ω. That is, our basic probability space is, say, $(\Omega,\mathfrak{F},P)$ and $\omega \in \Omega$. For each fixed t, $X(t,\omega)$ is a measurable function of ω. Now $N_n = N_n(\omega)$ can be written as

$$N_n(\omega) = \sum_{j=0}^{2^n-1} M_j(\omega)$$

$$M_j(\omega) = \begin{cases} 1 & \text{if } X(j2^{-n},\omega)\cdot X((j+1)2^{-n},\omega) < 0 , \\ 0 & \text{otherwise} . \end{cases}$$

But

$$\begin{aligned}\{\omega: M_j(\omega) = 1\} &= \{\omega: X(j2^{-n},\omega) < 0 < X((j+1)2^{-n},\omega\} \\ &\quad \cup\{\omega: X(j2^{-n},\omega) > 0 > X((j+1)2^{-n},\omega\} \\ &\quad \in \mathfrak{F}\end{aligned}$$

since $X(t,\omega)$ is a random variable for each t. Thus, clearly $N_n(\omega)$ is a random variable for each n. Since we have assumed that, with probability one, the sample functions are continuous, there is by Lemma 2.2.2 a measurable set A of $\mathfrak{F}$ such that $N_n(\omega) \longrightarrow N = N(\omega)$ for $\omega \in A$ and $P(A) = 1$. For $\omega \in \Omega\text{-}A$ let us define $N(\omega) = 0$. Then $N(\omega)$ as the probability one limit of a sequence of random variables is clearly a random variable itself. We may note that up until now N has only been defined with probability one.

We note here also that we should more properly talk about crossings by the sample functions or realizations of the process $X(t)$ although we shall usually use the less precise phrase "crossings by $X(t)$".

2.3 A Special Case - The Random Cosine Wave.

In almost no situation is it possible to actually obtain the distribution of the number of crossings or upcrossings. Since such results are possible for the so-called random cosine wave the simple ad hoc derivation will be given here.

Suppose we have a stochastic process $X(t)$ which can be expressed as

$$X(t) = A\cos(\omega t+\theta) \tag{2.3.1}$$

where ω is a fixed constant and A (the amplitude) and θ (the phase) are independent random variables A having a Rayleigh distribution, i.e. $P\{A > a\} = \exp(-a^2/2)$ for $a \geq 0$, and θ being uniformly distributed on $(0, 2\pi)$.

It is well-known that in this case $X(t)$ is a stationary normal process with zero mean and covariance function $r(\tau) = \cos\omega\tau$. The spectral distribution function has a single jump at the frequency ω. Thus the process and results to be obtained can be considered as approximations to the corresponding results for a stationary normal process with a very "narrow band" spectrum centered at ω.

Let $N_u(T)$ denote the number of upcrossings of the constant level a by $X(t)$ in the time interval $(0,T)$. From the nature of the process, the upcrossings (if any) always occur at points $2\pi/\omega$ apart. It is convenient to express T as $T = 2n\pi/\omega + 2\theta\pi/\omega$ where $0 \leq \theta < 1$ and n is a non-negative integer. In the interval $(0, 2n\pi/\omega)$ there will either be n upcrossings (if $A > a$) or none at all. Hence we need only consider in detail the interval $(2n\pi/\omega,T)$ or, by stationarity, we can consider the interval $(0,2\theta\pi/\omega)$. Consider a typical sample function of $X(t)$ as in Figure 2.3.1.

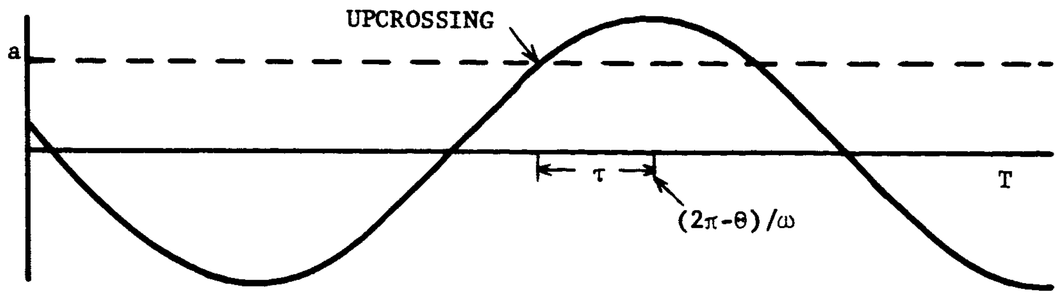


Figure 2.3.1

There can be at most one upcrossing in $(0,2\theta\pi/\omega)$ and this occurs only if $A > a$ and $\tau \leq (2\pi-\theta)/\omega \leq 2\theta\pi/\omega + \tau$ where τ, as in Figure 2.3.1, is the distance from an upcrossing to the next maximum of $X(t)$. But for any τ,

$$\begin{aligned} P\{A > a \text{ and } \tau \leq (2\pi-\theta)/\omega \leq 2\theta\pi/\omega + \tau\} &= P\{A > a\}.P\{0 \leq \theta \leq 2\theta\pi\} \\ &= \theta\exp(-a^2/2). \end{aligned}$$

Thus we have the complete result

Theorem 2.3.1: If $X(t)$ is a random cosine wave as defined above and $N_u(T)$ is the number of upcrossings of the level a in the interval $(0,T)$, then

$$\begin{aligned} P\{N_u(T) = 0\} &= 1 - e^{-a^2/2} \quad, \\ P\{N_u(T) = n\} &= (1-\theta)e^{-a^2/2} \quad, \\ P\{N_u(T) = n+1\} &= \theta e^{-a^2/2} \quad, \end{aligned}$$

where n and θ are defined by $T = 2n\pi/\omega + 2\theta\pi/\omega$, n being a non-negative integer and $0 \leq \theta < 1$.

Corollary 2.3.1:
$$\mathcal{E}\{N_u(T)\} = \frac{\omega T}{2\pi} e^{-a^2/2}$$

$$\mathrm{Var}\{N_u(T)\} = [n^2 + (2n+1)\theta - (n+\theta)^2 e^{-a^2/2}] e^{-a^2/2} .$$

For upcrossings of the zero level, i.e. $a = 0$, the variance reduces to $\theta(1-\theta)$.

CHAPTER III

THE MEAN NUMBER OF CURVE CROSSINGS

3.1 The Stationary Case.

The problem of obtaining the mean number of zeros or, more generally, the mean number of crossings of a level a by a stationary normal process has been considered by a number of authors beginning with the notable, if somewhat heuristic, work of Rice (1944). At about the same time Kac (1943) was able to obtain, by a mathematically rigorous method, the mean number of real roots of a polynomial with normally distributed coefficients. More recently these results have been extended to more general normal processes by Ivanov (1960), Bulinskaya (1961), Ito (1964) and Ylvisaker (1965). The two latter authors in fact give the complete solution to this problem. Specifically, if $X(t)$ is a zero mean, stationary normal process with continuous sample functions and N is the number of crossings of the level a by $X(t)$ for $t \in (0,1)$, then

$$(3.1.1) \qquad \mathcal{E}\{N\} = \frac{1}{\pi}[-r''(0)/r(0)]^{\frac{1}{2}} \exp[-a^2/(2r(0))] ,$$

where $r(\tau)$ is the covariance function of $X(t)$ and $r''(\tau) = \frac{d^2}{d\tau^2} r(\tau)$. The right-hand-side is to be interpreted as $+\infty$ if the second derivative of $r(\tau)$ at $\tau = 0$ does not exist. If the sample functions are not continuous with probability one, then the work of Belayev (1961) shows that $\mathcal{E}\{N\} = +\infty$ and hence all possible situations have been covered for level crossings by stationary normal processes. Under the same conditions given above, the mean number of upcrossings or down-crossings is given by $\frac{1}{2}\mathcal{E}\{N\}$.

Extending the problem in two directions, Leadbetter (1965) considered the situation where the fixed level a is replaced by a curve $a(t)$ and the normal process is either stationary or is the integral of a stationary process - a particular non-stationary case. Methods similar to those of Bulinskaya were used to obtain $\mathcal{E}\{N\}$ under rather weak sufficient conditions.

The first main result to be obtained here (Theorem 3.3.1) is also contained in a paper of Leadbetter and Cryer (1965a). It should be noted that the normality assumption is used in dealing with the joint distribution of X(t) and X'(t) (the quadratic mean derivative of X(t). That the results can be generalized to non-normal cases has recently been shown by Leadbetter (1966a) where certain conditions on such joint distributions are assumed.

3.2 The Non-stationary Case.

Suppose now we consider a general non-stationary normal process X(t) with mean $\mathcal{E}\{X(t)\} = m(t)$ and covariance $\text{cov}[X(t),X(s)] = \Gamma(t,s)$. Within the framework of non-stationary processes, curve crossings may easily be reduced to zero crossings since the number of crossings of the curve a(t) by X(t) is the same as the number of zero crossings by the non-stationary process X(t)-a(t). We note further that to obtain the mean number of zero crossings by X(t) for $t \in (a,b)$ it is sufficient to obtain the result for $t \in (0,1)$. To see this define a new process $\{Y(t):t\in(0,1)\}$ by $Y(t) = X(t(b-a))$. Then

$$\mathcal{E}\{Y(t)\} = m(t(b-a)) \text{ and } \text{cov}[Y(t),Y(s)] = \Gamma(t(b-a),s(b-a))$$

so continuity, differentiability, etc. of the mean and covariance of X(t) are equivalent to the corresponding properties of Y(t). Hence as long as our results are in terms of m, Γ and their derivatives, etc., there is no loss of generality in considering the problem of obtaining the mean number of zero crossings by X(t) for $t \in (0,1)$. The main result can now be stated and proved.

3.3 The Mean Number of Crossings.

We assume throughout this section that $\{X(t):t\in[0,1]\}$ is a normal process with $\mathcal{E}\{X(t)\} = m(t)$ and $\text{cov}[X(t),X(s)] = \Gamma(t,s)$. Let X'(t) denote the quadratic mean derivative of X(t).

<u>Theorem</u> 3.3.1: Suppose that $m(t)$ has a continuous derivative $m'(t)$ for $t\epsilon[0,1]$, that $\Gamma(t,s)$ has a mixed second partial derivative which is continuous at all diagonal points (t,t), $t\epsilon[0,1]$, and that the joint distribution of $X(t)$, $X'(t)$ is non-singular for each $t\epsilon[0,1]$. Then

$$(3.3.1) \qquad \mathcal{E}\{N\} = \int_0^1 \frac{\gamma(t)}{\sigma(t)}[1-\rho^2(t)]^{\frac{1}{2}} \phi[\frac{m(t)}{\sigma(t)}]\{2\phi[\eta(t)] + \eta(t)[2\Phi(\eta(t))-1]\}dt,$$

where

$$\sigma^2(t) = \Gamma(t,t),\ \gamma^2(t) = \frac{\partial^2\Gamma}{\partial t\partial s}\Big|_{s=t},\ \rho(t) = \frac{\partial\Gamma}{\partial s}\Big|_{s=t} /[\sigma(t)\gamma(t)],$$

$$\eta(t) = [\frac{m'(t)}{\gamma(t)} - \frac{\rho(t)m(t)}{\sigma(t)}][1-\rho^2(t)]^{-\frac{1}{2}} ,$$

$$\phi(x) = (2\pi)^{-\frac{1}{2}}e^{-x^2/2} \text{ , and } \Phi(x) = \int_{-\infty}^{x} \phi(u)du.$$

We note that $X'(t)$ exists by the assumptions on $\Gamma(t,s)$ and further $\sigma^2 = \text{var}[X(t)]$, $\gamma^2 = \text{var}[X'(t)]$, and $\rho = \text{cov}[X(t),X'(t)]$.

We further note that, under the conditions of the theorem, the work of Leadbetter (1966a) implies that the probability is zero that $X(t)$ will become tangential to the zero level somewhere in $0 \leq t \leq 1$ and thus the mean number of times $X(t)$ actually assumes the value zero is also given by the right-hand-side of (3.3.1).

The proof of this theorem will be given by a sequence of lemmas.

Let $\{Y_n(t): t\epsilon[0,1]\}$ be the piecewise linear process defined in (2.2) which approximates $X(t)$ and let N_n denote the number of zero crossings by $Y_n(t)$. By the Second Order Calculus Theorem of Loève (1960, p. 520) we know that, with probability one, the sample functions of $X(t)$ are continuous functions and further $X(k2^{-n}) \neq 0$, for $k=0,1,\dots 2^n$. Hence we may apply Lemma 2.2.2 and the monotone convergence theorem to obtain

<u>Lemma</u> 3.3.1: $\qquad \mathcal{E}\{N_n\} \longrightarrow \mathcal{E}\{N\}$ as $n \longrightarrow \infty$.

To evaluate $\mathcal{E}\{N_n\}$ we use a sequence of functions which, speaking loosely, approach a "Dirac delta function." Specifically, a sequence $\{\delta_n(x)\}$ of non-negative integrable functions is called a <u>delta-function sequence</u> if

$$\text{(i)} \quad \int_{-\infty}^{\infty} \delta_n(x)dx = 1 \qquad \text{for each } n$$

and for any $\epsilon > 0$ we have

$$\text{(ii)} \quad \lim_{n \to \infty} \int_{-\epsilon}^{\epsilon} \delta_n(x)dx = 1.$$

Let $Y_n'(t)$ denote the derivative of $Y_n(t)$ at points not of the form $k2^{-n}$, $k=0,1,\ldots 2^n$, the right hand derivative at $0, 2^{-n}, \ldots, (2^n-1)2^{-n}$ and the left hand derivative at $t = 1$. Then for any delta-function sequence $\{\delta_\nu(x)\}$ we have

<u>Lemma</u> 3.3.2: With probability one

$$N_n = \lim_{\nu \to \infty} \int_0^1 \delta_\nu[Y_n(t)]\,|Y_n'(t)|\,dt$$

and

$$\int_0^1 \delta_\nu[Y_n(t)]\,|Y_n'(t)|\,dt \leq 2^n .$$

Proof: Write $\alpha_k = k2^{-n}$, $Y_n(t) = A_k + B_k t$ for $t \in [\alpha_k, \alpha_{k+1})$ and $\beta_k = Y_n(\alpha_k)$. Then

$$\int_0^1 \delta_\nu[Y_n(t)]\,|Y_n'(t)|\,dt = \sum_{k=0}^{2^n-1} \int_{\alpha_k}^{\alpha_{k+1}} \delta_\nu[A_k+B_k t]\,|B_k|\,dt$$

$$(3.3.2) \qquad = \sum_{k=0}^{2^n-1} \left| \int_{\beta_k}^{\beta_{k+1}} \delta_\nu(x)dx \right|$$

With probability one, β_k is not zero for any $k=0,1,\ldots,2^n$. From the assumed delta-function properties it follows that if β_k and β_{k+1} have the same sign, the corresponding integral tends to zero as $\nu \to \infty$. If β_k and β_{k+1} have opposite signs this integral converges to ± 1. Thus if the interval (α_k, α_{k+1}) contains a zero of $Y_n(t)$ the corresponding term in the sum tends to 1 and otherwise it tends to zero. Hence the first part of the lemma is proved. The second part follows easily from (3.3.2) since each term in the sum is dominated by $\int_0^1 \delta_\nu(x)dx = 1$. Hence the lemma follows.

The results of this lemma enable us to apply dominated convergence and Fubini's Theorem for positive functions to obtain

$$\mathcal{E}\{N_n\} = \lim_{\nu \to \infty} \int_0^1 \mathcal{E}\{\delta_\nu[Y_n(t)]|Y_n'(t)|\}dt$$

$$(3.3.3) \qquad = \lim_{\nu \to \infty} \int_0^1 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |w|\delta_\nu(v)p_n(v,w)dvdwdt ,$$

where $p_n(v,w)$ is the bivariate normal density function for $Y_n(t), Y_n'(t)$. That is

$$p_n(v,w) = (2\pi D^{\frac{1}{2}})^{-1} \exp\{-[C(v-\alpha)^2 - 2B(v-\alpha)(w-\beta) + A(w-\beta^2)]/(2D)\}$$

in which

$$\alpha = \alpha_n(t) = \mathcal{E}\{Y_n(t)\} \quad , \qquad \beta = \beta_n(t) = \mathcal{E}\{Y_n'(t)\}$$

$$A = A_n(t) = \mathrm{Var}\{Y_n(t)\} \quad , \qquad C = C_n(t) = \mathrm{Var}\{Y_n'(t)\}$$

$$B = B_n(t) = \mathrm{Cov}[Y_n(t), Y_n(t)], \text{ and} \qquad D = D_n(t) = AC-B^2 .$$

We note that for <u>any</u> probability density function $f(x)$, we may obtain a delta-function sequence $\{\delta_\nu(x)\}$ by defining $\delta_\nu(x) = \nu f(\nu x)$. In particular if we take the normal density $f(x) = (2\pi)^{-\frac{1}{2}} e^{-x^2/2}$ we obtain from (3.3.3) after a simple change of variable

$$(3.3.4) \qquad \mathcal{E}\{N_n\} = \lim_{\nu \to \infty} (2\pi)^{-\frac{1}{2}} \int_0^1 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-v^2/2} |w| P_n(v/\nu, w)dvdwdt.$$

To simplify this expression and obtain its limit as $n \to \infty$ we need the following limits.

<u>Lemma</u> 3.3.3: The following are uniform limits in $0 \leq t \leq 1$ as $n \longrightarrow \infty$:

$$\text{(i)}\ \alpha_n(t) \longrightarrow m(t), \quad \text{(ii)}\ \beta_n(t) \longrightarrow m'(t), \quad \text{(iii)}\ A_n(t) \longrightarrow \Gamma(t,t),$$

$$\text{(iv)}\ B_n(t) \longrightarrow \Gamma_{01}(t,t), \text{ and (v) } C_n(t) \longrightarrow \Gamma_{11}(t,t)$$

where

$$\Gamma_{01}(t,t) = \left.\frac{\partial\Gamma(t,s)}{\partial s}\right|_{s=t} \text{ and } \Gamma_{11}(t,t) = \left.\frac{\partial^2\Gamma(t,s)}{\partial t\partial s}\right|_{s=t} .$$

That is, the moments and cross-moments of $Y_n(t)$, $Y_n'(t)$ converge uniformly to the corresponding moments of $X(t)$, $X'(t)$.

Proof: For notational convenience let us write $i_n = k_n(t)2^{-n}$ and $j_n = (k_n(t)+1)2^{-n}$, where as always $k_n(t)$ is the unique integer such that $k_n(t)2^{-n} \leq t < (k_n(t)+1)2^{-n}$. Thus, as $n \longrightarrow \infty$, i_n and j_n both tend to t uniformly.

Now by the definition of the $Y_n(t)$ process (equation 2.2.1) we have

$$\alpha_n(t) = m(i_n) + 2^n(t-i_n)[m(j_n)-m(i_n)] .$$

By the mean value theorem there is a number λ_n, $0 < \lambda_n < 2^{-n}$, such that

$$m(j_n) - m(i_n) = 2^{-n}\, m'(i_n+\lambda_n) .$$

Thus

$$\begin{aligned} |\alpha_n(t) - m(t)| &\leq |m(i_n) - m(t)| + |t - i_n||m'(i_n+\lambda_n)| \\ &\leq |m(i_n) - m(t)| + 2^{-n}K , \end{aligned}$$

where $K = \max\limits_{t\in[0,1]} |m'(t)| < \infty$. Hence, by the uniform continuity of $m(t)$, the uniform limit (i) is obtained.

Also by definition

$$\beta_n(t) = 2^n[m(j_n) - m(i_n)] ,$$

and thus

$$|\beta_n(t) - m'(t)| \leq |m'(t) - m'(i_n+\lambda_n)| .$$

Thus the uniform continuity of $m'(t)$ and the uniform convergence of $i_n + \lambda_n$ to t yields the limit (ii).

Again from the definition of $Y_n(t)$ we obtain

$$A_n(t) = 2^{2n}[(j_n-t)^2 \Gamma(i_n,i_n) + (t-i_n)^2 \Gamma(j_n,j_n) + 2(t-i_n)(j_n-t) \Gamma(i_n,j_n)]$$

$$= \Gamma(i_n,i_n) + 2^n(t-i_n)^2 [\Gamma_{10}(\mu_n,\mu_n) + \Gamma_{01}(\mu_n,\mu_n)]$$

$$+ 2^{n+1}(t-i_n)(j_n-t) \Gamma_{01}(i_n,\lambda_n)$$

where $i_n < \mu_n$, $\lambda_n < j_n$, using the mean value theorem twice. Now, by their definitions, $t - i_n \leq 2^{-n}$ and $j_n - t \leq 2^{-n}$. Hence

$$|A_n(t) - \Gamma(t,t)| \leq |\Gamma(t,t) - \Gamma(i_n,i_n)| + 2^{-n}|\Gamma_{10}(\mu_n,\mu_n) + \Gamma_{01}(\mu_n,\mu_n)|$$

$$+ 2^{-n+1}|\Gamma_{01}(i_n,\lambda_n)| .$$

But $\Gamma_{01}(t,s)$ and $\Gamma_{10}(t,s)$ are bounded for $0 \leq t, s \leq 1$ and $\Gamma(i_n,i_n) \longrightarrow \Gamma(t,t)$ uniformly. Therefore $A_n(t) \longrightarrow \Gamma(t,t)$ uniformly in $0 \leq t \leq 1$ and (iii) is proved.

In a similar manner

$$B_n(t) = 2^{2n}\{[(j_n-t) + (i_n-t)] \Gamma(i_n,j_n) - (j_n-t) \Gamma(i_n,i_n) + (t-i_n)\Gamma(j_n,j_n)\}$$

$$= [2^{n+1}(i_n-t) + 1] \Gamma_{01}(i_n,\mu_n) + 2^n(t-i_n)[\Gamma_{10}(\lambda_n,\lambda_n) + \Gamma_{01}(\lambda_n,\lambda_n)]$$

where $i_n < \mu_n$, $\lambda_n < j_n$.

Hence

$$|B_n(t) - \Gamma_{01}(t,t)| \leq |\Gamma_{01}(t,t) - \Gamma_{01}(i_n,\mu_n)| + 2^{n+1}(t-i_n)|\Gamma_{01}(\lambda_n,\lambda_n) - \Gamma_{01}(i_n,\mu_n)|$$

since $\Gamma_{01}(t,t) = \Gamma_{10}(t,t)$. Using $2^n(t-i_n) \leq 1$ and the uniform continuity of Γ_{01} we see that (iv) holds.

Finally we have

$$C_n(t) = 2^{2n}[\Gamma(i_n,i_n) - 2\Gamma(i_n,j_n) + \Gamma(j_n,j_n)] .$$

Let $\Psi(x) = \Gamma(x,j_n) - \Gamma(x,i_n)$. Then we can write

$$\begin{aligned} C_n(t) &= 2^{2n}[\Psi(j_n) - \Psi(i_n)] \\ &= 2^n \Psi'(\lambda_n) , \quad i_n < \lambda_n < j_n , \\ &= 2^n[\Gamma_{10}(\lambda_n,j_n) - \Gamma_{10}(\lambda_n,i_n)] , \\ &= \Gamma_{11}(\lambda_n,\mu_n) , \quad i_n < \mu_n < j_n . \end{aligned}$$

Thus the uniform continuity of $\Gamma_{11}(t,s)$ for $0 \leq t, s \leq 1$ shows that $C_n(t) \longrightarrow \Gamma_{11}(t,t)$ uniformly in $0 \leq t \leq 1$ and the proof of the lemma is completed.

We are now in a position to prove the theorem. Applying Lemma A.2 to the quadratic form in $p_n(v,w)$ we see that the integrand in (3.3.4) is dominated by $(2\pi D^{\frac{1}{2}})^{-1}|w|\exp\{-\frac{1}{2}[v^2+(w-\beta)^2/C]\}$ and converges to $(2\pi)^{-\frac{1}{2}}|w|e^{-v^2/2}\ p_n(0,w)$ as $\nu \longrightarrow \infty$. From the calculations of Lemma 3.3.3 it is clear that $\beta_n(t)$ and $C_n(t)$ are bounded functions of t for any n. By the same lemma

$$D_n(t) \longrightarrow \Gamma(t,t)\ \Gamma_{11}(t,t) - \Gamma_{01}^2(t,t)$$

uniformly and this is non-zero by the assumed non-singularity of the joint density of $X(t)$, $X'(t)$. Thus, at least for sufficiently large n, $D_n(t)$ is bounded away from zero for $0 \leq t \leq 1$. Hence, by dominated convergence,

$$\begin{aligned} \mathcal{E}\{N_n\} &= \int_0^1 \int_{-\infty}^{\infty} |w|p_n(0,w)dwdt \\ &= (2\pi)^{-1} \int_0^1 D^{-\frac{1}{2}} \int_{-\infty}^{\infty} |w|\exp\{-[C\alpha^2 + 2\alpha B(w-\beta) + A(w-\beta)^2]/(2D)\}dwdt . \end{aligned}$$

The change of variables $(A/D)^{\frac{1}{2}}(w-\beta) + \alpha B/(AD)^{\frac{1}{2}} = w'$ leads to

$$\begin{aligned} \mathcal{E}\{N_n\} &= (2\pi)^{-1} \int_0^1 (D^{\frac{1}{2}}/A)e^{-\alpha^2/(2A)} \int_{-\infty}^{\infty} |w + \delta|e^{-w^2/2}\ dwdt \\ &= (2/\pi)^{\frac{1}{2}} \int_0^1 (D^{\frac{1}{2}}/A)e^{-\alpha^2/(2A)}\{\phi(\delta) + \delta[\Phi(\delta)-\tfrac{1}{2}]\}\ dt \end{aligned}$$

where $\delta = \delta_n(t) = (A/D)^{\frac{1}{2}}(\beta-\alpha B/A)$.

Now using the limits of Lemma 3.3.3 and bounded convergence the required result follows.

The mean number of <u>upcrossings</u> or <u>downcrossings</u> of the zero level (or of a curve) may be obtained in a completely analogous fashion. The results may be stated as follows.

<u>Theorem</u> 3.3.2: Under the assumptions of Theorem 3.3.1 and with the same notation, the mean number of <u>upcrossings</u> of the zero level in [0,1] is

$$(3.3.5) \qquad \int_0^1 \frac{\gamma(t)}{\sigma(t)}[1-\rho^2(t)]^{\frac{1}{2}}\ \phi[\frac{m(t)}{\sigma(t)}]\{\phi[\eta(t)]+\eta(t)\Phi[\eta(t)]\}dt\ .$$

The mean number of <u>downcrossings</u> is given by

$$(3.3.6) \qquad \int_0^1 \frac{\gamma(t)}{\sigma(t)}[1-\rho^2(t)]^{\frac{1}{2}}\ \phi[\frac{m(t)}{\sigma(t)}]\{\phi[\eta(t)] - \eta(t)[1-\Phi[\eta(t)]]\}dt\ .$$

The details required for converting such results to those for curve crossings may now easily be stated.

<u>Theorem</u> 3.3.3: If $a(t)$ has a continuous derivative for $0 < t < 1$, then, under the same conditions as in Theorem 3.3.1, the mean number of crossings (upcrossings and downcrossings) of the curve $a(t)$ is given by (3.3.1) ((3.3.5), (3.3.6)) with $m(t)$ replaced by $m(t) - a(t)$ and $m'(t)$ replaced by $m'(t) - a'(t)$. Further the mean number of crossings (upcrossings, downcrossings) for t in [a,b] is obtained by intergrating t over this interval instead of [0,1].

When we are dealing with <u>stationary</u> processes these results simplify in the following manner.

<u>Corollary</u> 3.3.1: Suppose $X(t)$ is a stationary normal process with zero mean and covariance function $r(\tau)$. Let $a(t)$ have a continuous derivative $a'(t)$ for $t\epsilon[0,T]$. Then if $|r(\tau)| < 1$ for $\tau \neq 0$ and $r''(\tau)$ exists at $\tau = 0$, then the mean number of crossings of $a(t)$ for $t\epsilon[0,T]$ is given by

$$(3.3.7) \qquad \frac{1}{\sigma_o}\int_o^T \phi[a(t)/\sigma_o]\{2\sigma_2\phi[a'(t)/\sigma_2] + a'(t)(2\Phi[a'(t)/\sigma_2]-1)\}dt$$

where $\sigma_o^2 = r(0)$ and $\sigma_2^2 = -r''(0)$.

This special case is a result of Leadbetter (1965). Notice that σ_o and σ_2 are the only parameters of the process which appear in (3.3.7). Similar results follow for upcrossings and downcrossings.

<u>Corollary</u> 3.3.2: Under the conditions of Corollary 3.3.1, if a(t) is in fact a constant, a, then the mean number of crossings of a in [0,T] reduces to $\frac{T}{\pi}[-r''(0)/r(0)]^{\frac{1}{2}}\ e^{-a^2/[2r(0)]}$, the well known result for this case .

The conditions of Theorem 3.3.1 are, in fact, sufficient for the <u>finiteness</u> of $\mathcal{E}\{N\}$. The next theorem shows that a slight relaxation of the conditions leads to an infinite mean.

<u>Theorem</u> 3.3.3: Suppose all of the conditions of Theorem 3.3.1 hold except that the second mixed partial derivative of $\Gamma(t,s)$ does not exist at all diagonal points. Specifically we assume that, as $n \longrightarrow \infty$,

$$(3.3.8) \qquad C_n(t) = 2^{2n}[\Gamma(i_n,i_n) - 2\Gamma(i_n,i_n) + \Gamma(j_n,j_n)]$$

tends to infinity for t in a set S of positive Lebesgue measure. ($i_n = k_n(t)2^{-n}$, $j_n = i_n+2^{-n}$) Then $\mathcal{E}\{N\} = +\infty$.

Proof: From the proof of Theorem 3.3.1 we can still write

$$\mathcal{E}\{N_n\} = \lim_{\nu\longrightarrow\infty} (2\pi)^{-\frac{1}{2}}\int_o^1\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|w|e^{-v^2/2}\ p_u(v/\nu,w)dvdwd\tau$$

$$\geq \int_o^1\int_{-\infty}^{\infty}|w|p_n(0,w)dwd\tau \ ,$$

by Fatou's Lemma.

Explicitly

$$\int_{-\infty}^{\infty} |w|p_n(0,w)dw = (2\pi D^{\frac{1}{2}})^{-1} \int_{-\infty}^{\infty} |w|\exp\{-[\alpha^2C+2\alpha B(w-\beta)+A(w-\beta)^2]/(2D)\}dw$$

(3.3.10)
$$= \frac{D^{\frac{1}{2}}}{2\pi} \int_{-\infty}^{\infty} |w|\exp\{-[\alpha^2C+2\alpha B(D^{\frac{1}{2}}w-\beta)+A(D^{\frac{1}{2}}w-\beta)^2]/(2D)\}dw$$

by the change of variable $w = D^{\frac{1}{2}}w'$.

Now $A_n(t) \longrightarrow \Gamma(t,t)$, $B_n(t) \longrightarrow \Gamma_{01}(t,t)$, $\alpha_n(t) \longrightarrow m(t)$, $\beta_n(t) \longrightarrow m'(t)$, but $C_n(t) \longrightarrow +\infty$ for $t\in S$. Thus

$$\frac{C}{D} = \frac{C}{AC-B^2} = \frac{1}{A-B^2/C} \longrightarrow \frac{1}{\Gamma(t,t)}$$

$$\frac{\alpha B}{D^{\frac{1}{2}}} \longrightarrow 0 \quad \text{and} \quad \frac{A}{D^{\frac{1}{2}}} \longrightarrow 0$$

Hence from (3.3.10) $\int_{-\infty}^{\infty} |w|p_n(0,w)dw \longrightarrow +\infty$ as $n \longrightarrow \infty$ for $t\in S$. Therefore another application of Fatou's Lemma to (3.3.9) yields $\mathcal{E}\{N_n\} \longrightarrow +\infty$ and thus the theorem follows.

The condition that $C_n(t)$ of (3.3.8) tend to infinity is not very easy to check in terms of a given covariance function. A somewhat simpler condition can be given as follows.

<u>Corollary</u> 3.3.3: Let $\Delta(h,t) = h^{-2}[\Gamma(t+h,t+h) - 2\Gamma(t,t+h) + \Gamma(t,t)]$. If $\Delta(h,t) \longrightarrow \infty$ as $h \longrightarrow 0$ uniformly for t in some subinterval I of $(0,1)$, then $C_n(t) \longrightarrow \infty$ as required in the theorem.

The proof is easy since $i_n \longrightarrow t$ uniformly in t as $n \longrightarrow \infty$ and $j_n = i_n + h$ if we take $h = 2^{-n}$.

3.4 <u>Examples</u>

(i) The integrated Wiener process.

Let $\{W(t):t\in[0,\infty)\}$ denote the separable Wiener process and let $X(t) = \int_0^t W(u)du$. Since the sample functions of the Wiener process are known to be almost surely continuous we can take $X(t)$ as a sample function integral. Now the Wiener process is defined as a normal process with zero mean and covariance function $\mathcal{E}\{W(t)W(s)\} = \min(t,s)$. Hence it is easy to show that $X(t)$ has zero mean and covariance $\Gamma(t,s) = \frac{s^2}{6}(3t-s)$ for $0 \leq s \leq t$; see Parzen (1962), for example. The derivatives required for Theorem 3.3.1 are readily found to be $\Gamma_{01}(t,t) = t^2/2$ and $\Gamma_{11}(t,t) = t$. Thus $\sigma(t) = (t^3/3)^{\frac{1}{2}}$, $\gamma(t) = t^{\frac{1}{2}}$ and for $t > 0$, $\rho(t) = (3/4)^{\frac{1}{2}}$. We note that the joint distribution of $X(t)$, $X'(t)$ is singular at $t = 0$, the variances being zero, but the assumptions of the theorem <u>are</u> satisfied for t in any interval $[a,b]$ with $a > 0$. Hence the mean number of zero crossings by the integrated Wiener process for $t \in [a,b]$ is given by

$$\int_a^b \frac{3^{\frac{1}{2}}}{t}(1-3/4)^{\frac{1}{2}}\, \phi(0)[2\phi(0)]dt = \frac{3^{\frac{1}{2}}}{2\pi} \log \frac{b}{a}$$

(ii) Crossings of a "linear ramp" by a stationary process.

Suppose $X(t)$ is a stationary normal process with zero mean and covariance $r(\tau)$ satisfying the assumptions of Corollary 3.3.1. The mean number of crossings of the "linear ramp" $a(t) = a+bt$ $(b \neq 0)$ for t in the interval $(0,T)$ is given by

$$\frac{1}{\sigma_o}\{2\sigma_2\phi(b/\sigma_2) + b[2\Phi(b/\sigma_2)-1]\}\cdot\int_o^T \phi[(a+bt)/\sigma_o]dt$$

$$= \{\frac{2\sigma_2}{b}\, \phi(b/\sigma_2) + [2\Phi(b/\sigma_2)-1]\}\cdot\{\Phi[(a+bT)/\sigma_o] - \Phi(a/\sigma_o)\} ,$$

a result given by Leadbetter (1965).

(iii) Reliability - a numerical example.

As stated is the introductory chapter, in analyzing complex systems from a reliability point of view it is sometimes convenient to regard some pertinent time-dependent output, such as angular error in missile attitude, as a stochastic process. For good performance we would like to have this process remain within certain bounds during the mission time. But perhaps certain time periods are more critical than others (for example the "lift-off" phase); then the bounds must account for such periods and must vary with time.

If we are considering a missile system then a reasonable upper bound for a zero mean output $X(t)$ which is critical at lift-off would be

$$a(t) = a - \frac{b}{t+c}$$

where $a - b/c > 0$ and $a,b > 0$.

If we can reasonably assume that $X(t)$ is a stationary normal process then $M(T)$, the mean number of crossings of this bound in a mission of length T, can be computed by means of Corollary 3.3.1. From a performance point-of-view the smaller $M(T)$ the better the system.

As an example $M(T)$ was computed by numerical integration for particular values of a,b,c and for a stationary process with parameters σ_o, σ_2. Considered as a function of T, $M(T)$ is graphed in Figure 3.4.1.

MEAN NUMBER OF CROSSINGS OF THE CURVE $a(t) = a - \frac{b}{t+c}$

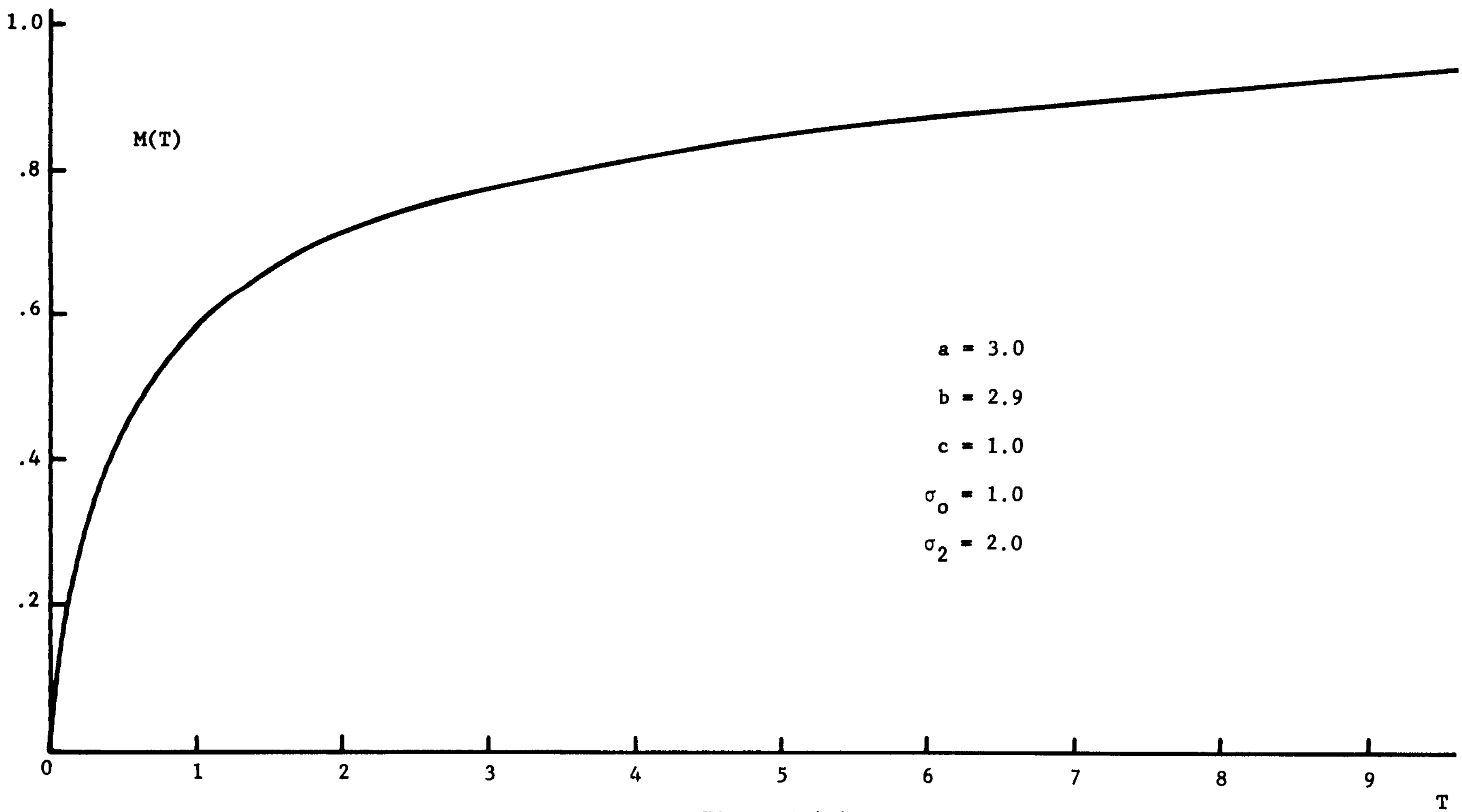


Figure 3.4.1

CHAPTER IV

THE VARIANCE OF THE NUMBER OF CROSSINGS

4.1 Introduction.

In the preceding chapter the mean number of crossings of a curve by a non-stationary normal process was obtained under rather general conditions and it was noted that, in the stationary case, complete results are available. However results on the variance of the number of crossings are more scattered and are all concerned with stationary normal processes. The classical work of Rice (1944) implicitly contains the formula for the second moment but no conditions for its validity are given. Similar heuristic methods were employed by Miller and Freund (1956) and Steinberg, et al. (1955) obtained the variance of the number of zero crossings for a particular stationary normal process. The first derivation for a somewhat general situation seems to occur in a footnote of a paper by Volkonski and Rozanov (1961) where it is assumed that the covariance function has a sixth derivative at the origin. Recently Cramer and Leadbetter (1965) have obtained the formulae for the factorial moments of the number of upcrossings under the condition that the process have, with probability one, a continuous sample derivative. Their work however does not deal with the finiteness of the moments. The main result we obtain here, Theorem 4.2.1, is essentially that announced in Leadbetter and Cryer (1965b).

In Section 2, the second moment of the number of upcrossings of the level a by a stationary normal process will be obtained. The conditions given are sufficient for finiteness of the second moment and, as will be shown by an example, are in fact very close to the necessary conditions. In Section 3 it is shown how this result can be used to obtain the corresponding second moment for the total number of crossings. Section 4 deals with the covariance of the number of upcrossings in two intervals. In the remaining sections the asymptotic form of the variance as the length of the time interval tends to infinity is investigated and some numerical calculations are presented.

4.2 The Variance.

Throughout the remainder of this chapter we assume that $X(t)$ is a stationary normal process which has, with probability one, continuous sample functions. The time parameter t is contained in an appropriate index set-either $[0,1]$, $[0,T]$, or $[0,\infty)$. We assume $\mathcal{E}\{X(t)\} = 0$ and denote the (continuous) covariance function by $r(\tau) = \mathcal{E}\{X(t)\, X(t+\tau)\}$. The corresponding (integrated) spectrum $F(\lambda)$ satisfies

$$(4.2.1) \qquad r(\tau) = \int_0^\infty \cos\lambda\tau \, dF(\lambda) \quad .$$

In considering the number of crossings (or upcrossings) of the level a by a stationary process we can assume the variance, $r(0)$, is unity since the number of crossings of a by $X(t)$ is the same as the number of crossings of $a/[r(0)]^{\frac{1}{2}}$ by $X(t)/[r(0)]^{\frac{1}{2}}$, a process with unit variance. Hence we suppose $r(0) = 1$ with no loss of generality.

Let N_u denote the number of upcrossings of the level a by the process $X(t)$ for $t \in [0,1]$ and let $-r''(0) = \lambda_2 = \int_0^\infty \lambda^2 dF(\lambda)$. The main result may then be stated as follows.

Theorem 4.2.1: Suppose the spectral distribution $F(\lambda)$ has a continuous component and that the covariance function $r(\tau)$ has a second derivative $r''(\tau)$ which, for all sufficiently small τ, satisfies

$$(4.2.2) \qquad \lambda_2 + r''(\tau) \leq \Psi(\tau)$$

where $\Psi(\tau)/\tau$ is integrable over $[0,1]$ and $\Psi(\tau)$ decreases as τ decreases to zero. Then the second moment of N_u is finite and is given by

$$(4.2.3) \quad \mathcal{E}\{N_u^2\} = \mathcal{E}\{N_u\} + \int_0^1\int_0^1 \int_0^\infty\int_0^\infty xy\, p_{t-s}(a,a,x,y)dxdydsdt \ ,$$

where $p_\tau(u,v,x,y)$ is the four-dimensional normal density for the variables $X(0)$, $X(\tau)$, $X'(0)$, $X'(\tau)$, $X'(\tau)$ denoting the quadratic mean derivative of $X(\tau)$.

The statement of the theorem as it stands is convenient for theoretical purposes. From a practical (computational) standpoint, however, the right-hand side of (4.2.3) may be made somewhat more explicit, especially in the zero level case, i.e. $a=0$. Specifically we may write

$$(4.2.4) \qquad p_\tau(u,v,x,y) = (2\pi)^{-2}\,|\Sigma|^{-\frac{1}{2}} \exp[-(u,v,x,y)\,\Sigma^{-1}(u,v,x,y)'/2]$$

where the covariance matrix, $\Sigma = \Sigma(\tau)$, is given as

$$(4.2.5) \qquad \Sigma = \begin{bmatrix} 1 & r(\tau) & 0 & r'(\tau) \\ r(\tau) & 1 & -r'(\tau) & 0 \\ 0 & -r'(\tau) & \lambda_2 & -r''(\tau) \\ r'(\tau) & 0 & -r''(\tau) & \lambda_2 \end{bmatrix}$$

When $a=0$ Equation (4.2.3) may be evaluated using Lemma A.3 of the Appendix to yield

$$(4.2.6) \qquad \mathcal{E}\{N_u^2\} = \frac{\lambda_2^{\frac{1}{2}}}{2\pi} + \frac{1}{2\pi^2}\int_0^1 (1-\tau)(\Sigma_{33}^2-\Sigma_{34}^2)^{\frac{1}{2}}[1-r^2(\tau)]^{-3/2}(1-\Delta\cot^{-1}\Delta)d\tau \ ,$$

where Σ_{ij} is the cofactor of the ij-th element of Σ and $\Delta = \Sigma_{34}(\Sigma_{33}^2-\Sigma_{34}^2)^{-\frac{1}{2}}$, the dependence of Σ_{ij} and Δ on τ being suppressed.

The proof of the theorem is quite long and will be obtained from the several lemmas which follow.

Let N_n denote the number of upcrossings of the level a by the piecewise linear process $Y_n(t)$ as defined in Chapters II and III.

Lemma 4.2.1: If $\{\delta_m(x)\}$ is a delta-function sequence (defined in 3.3) and $\sigma(x)$ is defined as

$$\sigma(x) = \begin{cases} x & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases} \ ,$$

then, with probability one,

$$N_n = \lim_{m \to \infty} \int_o^1 \delta_m[Y_n(t)-a]\ \sigma[Y_n'(t)]dt \ .$$

Proof: Write $\alpha_k = k2^{-n}$, $Y_n(t) = A_k + B_k t$ for $t \in [\alpha_k, \alpha_{k+1})$ and $\beta_k = Y_n(\alpha_k) - a$.

Then

$$\int_o^1 \delta_m[Y_n(t)-a]\ \sigma[Y_n'(t)]dt = \sum_{k=0}^{2^n-1} \int_{\alpha_k}^{\alpha_{k+1}} \delta_m[A_k+B_k t-a]\ \sigma(B_k)dt$$

$$= \Sigma' \int_{\beta_k}^{\beta_{k+1}} \delta_m(x)dx$$

where the last summation is over those k such that $B_k > 0$. From the properties of the delta-function sequence this converges to N_n as in the proof of Lemma 3.3.2.

Now consider the unit square $\{(s,t)\colon\ 0 \leq s,\ t \leq 1\}$. Let n be a fixed positive integer. If for some integer k, $k2^{-n} \leq s,\ t \leq (k+1)2^{-n}$ we say that s and t are in the <u>same</u> interval. If $k2^{-n} \leq s < (k+1)2^{-n} \leq t < (k+2)2^{-n}$ or if $k2^{-n} \leq t < (k+1)2^{-n} \leq s < (k+2)2^{-n}$ we say that s and t are in <u>adjacent</u> intervals. If s and t are <u>not</u> in the same interval nor in adjacent intervals we say that they are in separated intervals and write S_n for the subset where s and t are in separated intervals. With this notation we have

<u>Lemma</u> 4.2.2: With probability one

$$N_n^2 = N_n + \lim_{m \to \infty} \int_{S_n}\!\!\int \delta_m[Y_n(t)-a]\ \delta_m[Y_n(s)-a]\ \sigma[Y_n'(t)]\ \sigma[Y_n'(s)]dsdt \ ,$$

and

$$\int_{S_n}\!\!\int \delta_m[Y_n(t)-a]\ \delta_m[Y_n(s)-a]\ \sigma[Y_n'(t)]\ \sigma[Y_n'(s)]dsdt \leq 2^{2n} \ .$$

Proof: Again let $\alpha_k = k2^{-n}$. Then from Lemma 4.2.1

$$N_n^2 = \lim_{m\to\infty} \left[\sum_{j=0}^{2^n-1} \sum_{i=0}^{2^n-1} \int_{\alpha_j}^{\alpha_{j+1}} \int_{\alpha_i}^{\alpha_{i+1}} \delta_m[Y_n(t)-a]\delta_m[Y_n(s)-a]\sigma[Y_n'(t)]\sigma[Y_n'(s)]dsdt \right]$$

$$= \lim_{m\to\infty} \sum_{i=0}^{2^n-1} \left\{ \int_{\alpha_i}^{\alpha_{i+1}} \delta_m[Y_n(t)-a]\sigma[Y_n'(t)]dt \right\}^2$$

$$+ \lim_{m\to\infty} 2 \sum_{i=0}^{2^n-2} \left[\int_{\alpha_i}^{\alpha_{i+1}} \delta_m[Y_n(t)-a]\sigma[Y_n'(t)]dt \cdot \int_{\alpha_{i+1}}^{\alpha_{i+2}} \delta_m[Y_n(s)-a]\sigma[Y_n'(s)]ds \right]$$

$$+ \lim_{m\to\infty} \int\int_{S_n} \delta_m[Y_n(t)-a]\delta_m[Y_n(s)-a]\sigma[Y_n'(t)]\sigma[Y_n'(s)]dsdt \ .$$

The first term is N_n as in the proof of Lemmas 3.3.2 and 4.2.1 and hence we need only show that the second term is zero.

Consider

$$\int_{\alpha_i}^{\alpha_{i+1}} \delta_m[Y_n(t)-a]\sigma[Y_n'(t)]dt \cdot \int_{\alpha_{i+1}}^{\alpha_{i+2}} \delta_m[Y_n(s)-a]\sigma[Y_n'(s)]ds \ .$$

By the construction of the process $Y_n(t)$ it is impossible to have an <u>up</u>crossing of the level a in each of two adjacent intervals. Therefore at least one of the above factors tends to zero as $m \to \infty$ and hence the entire second term is zero in the limit.

The last part of the lemma follows as in Lemmas 3.3.2 and 4.2.1.

Using Lemma 4.2.2, the dominated convergence theorem and Fubini's theorem for positive functions we immediately obtain

Lemma 4.2.3:

$$\mathcal{E}\{N_n^2\} = \mathcal{E}\{N_n\} + \lim_{m\to\infty} \int_{S_n}\int \mathcal{E}\{\delta_m[Y_n(t)-a]\delta_m[Y_n(s)-a]\sigma[Y_n'(t)]\sigma[Y_n'(s)]\}dsdt.$$

To evaluate the second term on the right we need the results of the next two lemmas.

Let $\Sigma_n = \Sigma_n(t,s)$ denote the covariance matrix for the variables $Y_n(t)$, $Y_n(s)$, $Y_n'(t)$, $Y_n'(s)$.

Lemma 4.2.4: For $(s,t) \in S_n$ the joint (normal) distribution of $Y_n(t)$, $Y_n(s)$, $Y_n'(t)$, $Y_n'(s)$ is non-singular for each n and further there is a positive constant c such that $|\Sigma_n(t,s)| \geq c > 0$ uniformly for $(s,t) \in S_n$.

Proof: Let $i = k_n(t)$, $j = k_n(s)$. Then

$$Y_n(t) = X(i2^{-n}) + 2^n(t-i2^{-n})[X((i+1)2^{-n}) - X(i2^{-n})]$$

$$Y_n'(t) = 2^n[X((i+1)2^{-n}) - X(i2^{-n})]$$

with similar expressions for $Y_n(s)$, $Y_n'(s)$ (replace t by s, i by j). Hence the vector $\underline{y}' = (Y_n(t), Y_n(s), Y_n'(t), Y_n'(s))$ may be written as a linear transformation of the vector $\underline{x}' = (X(i2^{-n}), X((i+1)2^{-n}), X(j2^{-n}), X((j+1)2^{-n}))$, viz. $\underline{y} = M\,\underline{x}$ where M is given as

$$M = \begin{bmatrix} 1-2^nt+i & 2^nt-i & 0 & 0 \\ 0 & 0 & 1-2^ns+j & 2^ns-j \\ -2^n & 2^n & 0 & 0 \\ 0 & 0 & -2^n & 2^n \end{bmatrix}$$

Since $(s,t) \in S_n$ the integers i, i+1, j, j+1 are all different and thus by Lemma A.1 the joint density of $\underline{x}$ is non-singular. But the determinant of M is easily found to be $|M| = 2^{2n}$ and therefore the joint distribution of $\underline{y}$ is always non-singular. Further as (s,t) varies over S_n, the determinant of the covariance matrix $\mathrm{Cov}(\underline{x})$ takes on only a finite number of values, all of them strictly positive. But $\mathrm{Cov}(\underline{y}) = M\,\mathrm{Cov}(\underline{x})\,M'$ and thus $|\mathrm{Cov}(\underline{y})| = 2^{4n}\,|\mathrm{Cov}(\underline{x})|$ and the existence of the constant c is established.

Lemma 4.2.5: As $n \to \infty$ the following limits hold uniformly for $0 \le s,\ t \le 1$.

$$\text{(i)}\quad \mathrm{Cov}[Y_n(t),\ Y_n(s)] \to r(t-s)\ ,$$

$$\text{(ii)}\quad \mathrm{Cov}[Y_n'(t),\ Y_n(s)] \to r'(t-s)\ ,$$

$$\text{(iii)}\quad \mathrm{Cov}[Y_n'(t),\ Y_n'(s)] \to -r''(t-s).$$

Proof: From the definition of $Y_n(t)$ we have

$$\begin{aligned}
&\mathrm{Cov}[Y_n(t),\ Y_n(s)]\\
&\quad = [(1-2^n t+i_n)(1-2^n s+j_n) + (2^n t-i_n)(2^n s-j_n)]\ r((i_n-j_n)2^{-n})\\
&\quad\quad + (1-2^n t+i_n)(2^n s-j_n)r((i_n-j_n-1)2^{-n}) + (1-2^n s+j_n)(2^n t-i_n)r((i_n-j_n+1)2^{-n})
\end{aligned}$$

where $i_n = k_n(t)$ and $j_n = k_n(s)$.

Expanding the covariance function $r(\tau)$ about the point $t-s$ leads to

$$\begin{aligned}
&\mathrm{Cov}[Y_n(t),\ Y_n(s)] - r(t-s)\\
&\quad = 2^{-n}[(1-2^n t+i_n)(1-2^n s+j_n) + (2^n s-j_n)(2^n t-i_n)][i_n-j_n-2^n(t-s)]\ r'(\theta_1)\\
&\quad\quad + 2^{-n}(1-2^n t+i_n)(2^n s-j_n)(i_n-j_n-1-2^n(t-s))\ r'(\theta_2)\\
&\quad\quad + 2^{-n}(1-2^n s+j_n)(2^n t-i_n)(i_n-j_n+1-2^n(t-s))\ r'(\theta_3)\ ,
\end{aligned}$$

where

$$\begin{aligned}
&0 < |t-s-\theta_1| < 2^{-n}|i_n-j_n-2^n(t-s)| < 2\cdot 2^{-n}\\
&0 < |t-s-\theta_2| < 2^{-n}|i_n-j_n-1-2^n(t-s)| < 2\cdot 2^{-n} \qquad (4.2.7)\\
&0 < |t-s-\theta_3| < 2^{-n}|i_n-j_n+1-2^n(t-s)| < 2\cdot 2^{-n}\ .
\end{aligned}$$

By definition of i_n, j_n the quantities $|1-2^n t+i_n|$, $|1-2^n s+j_n|$, $|2^n t-i_n|$ and $|2^n s-j_n|$ are all bounded by 1 and thus

$$|\mathrm{Cov}[Y_n(t),\ Y_n(s)] - r(t-s)| \leq (4|r'(\theta_1)| + 2|r'(\theta_2)| + 2|r'(\theta_3)|)\ 2^{-n}\ .$$

Since $r'(\tau)$ is uniformly continuous and bounded for $0 \leq \tau \leq 1$ the required uniform limit (i) is obtained.

Again by definition we have

$$\begin{aligned}&\mathrm{Cov}[Y_n'(t),\ Y_n(s)]\\ &= 2^n\{(1-2^n s+j_n)[r((i_n-j_n+1)2^{-n}) - r((i_n-j_n)\ 2^{-n})]\\ &\qquad + (2^n s-j_n)[r((i_n-j_n)\ 2^{-n}) - r((i_n-j_n-1)\ 2^{-n})]\}\end{aligned}$$

Using three term expansions we find

$$\begin{aligned}&\mathrm{Cov}[Y_n'(t),\ Y_n(s)] - r'(t-s)\\ &= 2^{-n-1}\{(1-2^n s+j_n)[(i_n-j_n+1-2^n(t-s))^2\ r''(\theta_3) - (i_n-j_n-2^n(t-s))^2\ r''(\theta_1)]\\ &\qquad + (2^n s-j_n)[(i_n-j_n-2^n(t-s))^2\ r''(\theta_1) - (i_n-j_n-1-2^n(t-s))^2\ r''(\theta_2)]\}\end{aligned}$$

where the (new) θ_i satisfy (4.2.7). Hence

$$|\mathrm{Cov}[Y_n'(t),\ Y_n(s)] - r'(t-s)| \leq (8|r''(\theta_1)| + 4|r''(\theta_2)| + 4|r''(\theta_3)|)2^{-n-1}$$

and since $r''(\tau)$ is also uniformly continuous and bounded for $0 \leq \tau \leq 1$ the desired result (ii) holds.

Finally we have

$$\mathrm{Cov}[Y_n'(t),\ Y_n'(s)] = 2^{2n}[2r((i_n-j_n)2^{-n}) - r((i_n-j_n-1)2^{-n}) - r((i_n-j_n+1)2^{-n}]$$

Again using three term expansions we obtain

$$\mathrm{Cov}[Y_n'(t),\ Y_n'(s)] = -\tfrac{1}{2}[r''(\theta_1) + r''(\theta_2)]$$

where

$$|2^{-n}(i_n-j_n) - \theta_1| < 2^{-n},\ |2^{-n}(i_n-j_n) - \theta_2| < 2^{-n}\ .$$

Now by the definition of i_n and j_n, $(i_n - j_n)2^{-n} \longrightarrow t-s$ uniformly in (t,s) as $n \longrightarrow \infty$. Therefore θ_1 and θ_2 each converge to $t-s$ uniformly and by the uniform continuity of r'', (iii) is proved.

Lemma 4.2.6:

$$\mathcal{E}\{N_n^2\} = \mathcal{E}\{N_n\} + \int\int_{S_n} \int_0^\infty \int_0^\infty xy\, p_{n,t,s}(a,a,x,y)dxdydsdt \, , \tag{4.2.8}$$

where $p_{n,t,s}(u,v,x,y)$ is the four-dimensional normal density for $Y_n(t)$, $Y_n(s)$, $Y_n'(t)$, $Y_n'(s)$.

Proof: We first note that for $(s,t) \in S_n$ we have

$$\begin{aligned} &\mathcal{E}\{\delta_m[Y_n(t)-a]\ \delta_m[Y_n(s)-a]\ \sigma[Y_n'(t)]\ \sigma[Y_n'(s)]\} \\ &= \int_0^\infty \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty xy\ \delta_m(u-a)\ \delta_m(v-a)\ p_{n,t,s}(u,v,x,y)dudvdxdy \, . \end{aligned} \tag{4.2.9}$$

Explicitly

$$p_{n,t,s}(u,v,x,y) = (2\pi)^{-2}\ |\Sigma_n|^{-\frac{1}{2}} \exp[-\tfrac{1}{2}(u,v,x,y)\ \Sigma_n^{-1}(u,v,x,y)']$$

and by taking, as in Chapter III, $\delta_m(x) = m(2\pi)^{-\frac{1}{2}} \exp(-m^2x^2/2)$ the right-hand side of (4.2.9) becomes

$$m^2(2\pi)^{-3}\ |\Sigma_n|^{-\frac{1}{2}} \int_0^\infty \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty xy\ \exp\{-\tfrac{1}{2}[m^2(u-a)^2+m^2(v-a)^2+(u,v,x,y)\Sigma_n^{-1}(u,v,x,y)']\}\ dudvdxdy$$

$$= (2\pi)^{-3}|\Sigma_n|^{-\frac{1}{2}} \int_0^\infty \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty xy\ \exp\{-\tfrac{1}{2}[u^2+v^2+(a+u/m,a+v/m,x,y)\Sigma_n^{-1}(a+u/m,a+v/m,x,y)']\}\ dudvdxdy$$

Now by Lemma 4.2.4 we know that $|\Sigma_n| = |\Sigma_n(t,s)|$ is bounded away from zero uniformly for $(s,t) \in S_n$. Further an application of Lemma A.2 shows that

$$\begin{aligned} &\exp[-\tfrac{1}{2}(u,v,x,y)\ \Sigma_n^{-1}(u,v,x,y)'] \\ &\leq \exp[-\tfrac{1}{2}(x,y)\ \Lambda_3^{-1}(x,y)'] \end{aligned}$$

where Λ_3 is the covariance matrix of $Y_n'(t)$, $Y_n'(s)$. As a corollary to Lemma 4.2.4, Λ_3 is non-singular for $(s,t) \in S_n$ and indeed by Lemma 4.2.5 the elements of Λ_3^{-1} are bounded functions of s and t, the diagonal elements being bounded away from zero when $(s,t) \in S_n$.

Hence by dominated convergence we obtain

$$\lim_{m\to\infty} \int\int_{S_n} \mathcal{E}\{\delta_m[Y_n(t)-a]\delta_m[Y_n(s)-a]\sigma[Y_n'(t)]\sigma[Y_n'(s)]\}dsdt$$

$$= \int\int_{S_n}\int_0^\infty\int_0^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty (2\pi)^{-1} xy \lim_{m\to\infty} e^{-\frac{1}{2}u^2+\frac{1}{2}v^2} p_{n,t,s}(a+u/m,a+v/m,x,y)$$

$$dudvdxdydsdt$$

$$= \int\int_{S_n}\int_0^\infty\int_0^\infty xy\, p_{n,t,s}(a,a,x,y)dxdydsdt ,$$

and thus the result is obtained by an application of Lemma 4.2.3.

In order to obtain the limit of the second term on the right of (4.2.8) we must investigate the behavior of $\chi_n \cdot \int_0^\infty\int_0^\infty xy\, p_{n,t,s}(a,a,x,y)dxdy$ where χ_n is the indicator function of the set S_n, for $t-s = \tau$ in a neighborhood of zero. This will be done in several steps.

<u>Lemma</u> 4.2.7: For $(s,t) \in S_n$ we have

$$(4.2.9) \quad \int_0^\infty\int_0^\infty xy\, p_{n,t,s}(a,a,x,y)dxdy \le (2\pi|\Lambda_1|^{\frac{1}{2}})^{-1}\{M_{44}|\Lambda_1|^{-1}+\mu_1^2\}^{\frac{1}{2}}\cdot\{M_{33}|\Lambda_1|^{-1}+\mu_2^2\}^{\frac{1}{2}},$$

where $\Sigma_n(t,s)$ is partitioned into 2 x 2 submatrices as $\begin{bmatrix}\Lambda_1 & \Lambda_2\\ \Lambda_2 & \Lambda_3\end{bmatrix}$, M_{ij} denotes the cofactor of the ij-th element of $\Sigma_n(t,s)$ and $(\mu_1,\mu_2)' = \Lambda_2' \Lambda_1^{-1}(a,a)'$.

Proof: Let $\underline{x}' = (x,y)$, $\underline{a}' = (a,a)$, $\underline{\mu}' = (\mu_1,\mu_2)$ and $\Lambda = \Lambda_3-\Lambda_2'\Lambda_1^{-1}\Lambda_2$. Then we can write (cf. the conditional density of $Y_n'(t)$, $Y_n'(s)$ given $Y_n(t) = Y_n(s) = a$; see Anderson (1958), for example.)

$$p_{n,t,s}(a,a,x,y) = \frac{\exp(-\frac{1}{2}\underline{a}\ \Lambda_1^{-1}\underline{a}')}{2\pi\ |\Lambda_1|^{\frac{1}{2}}} \cdot \frac{1}{2\pi\ |\Lambda|^{\frac{1}{2}}} \exp[-\tfrac{1}{2}(\underline{x}-\underline{\mu})'\ \Lambda^{-1}\ (\underline{x}-\underline{\mu})\ ] ,$$

where we use the fact that $|\Sigma_n| = |\Lambda_1|\ |\Lambda\ |$.

We notice that for $(s,t) \in S_n$, $\Sigma_n(t,s)$ is positive definite and thus so is Λ_1. Hence the factor $\exp(-\frac{1}{2}\underline{a}\ \Lambda_1^{-1}\underline{a}')$ never exceeds unity.

By application of Schwarz' inequality we find

$$\int_0^\infty\int_0^\infty xy\ p_{n,t,s}(a,a,x,y)dxdy$$

$$\leq\ (2\pi|\Lambda_1|^{\frac{1}{2}})^{-1}\ \{(2\pi|\Lambda|^{\frac{1}{2}})^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^2\ \exp[-\tfrac{1}{2}(\underline{x}-\underline{\mu})'\Lambda^{-1}(\underline{x}-\underline{\mu})\ ]dxdy\}^{\frac{1}{2}}$$

$$\cdot\{(2\pi|\Lambda|^{\frac{1}{2}})^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y^2\ \exp[-\tfrac{1}{2}(\underline{x}-\underline{\mu})'\Lambda^{-1}(\underline{x}-\underline{\mu})\ ]dxdy\}^{\frac{1}{2}}$$

$$=\ (2\pi|\Lambda_1|^{\frac{1}{2}})^{-1}\ \{\Lambda_{11}+\mu_1^2\}^{\frac{1}{2}}\cdot\{\Lambda_{22}+\mu_2^2\}^{\frac{1}{2}}\ ,$$

where Λ_{ij} denotes the ij-th element of Λ.

Now let Σ_n^{-1} be also partitioned into 2 x 2 submatrices $\begin{bmatrix} B_1 & B_2 \\ B_2' & B_3 \end{bmatrix}$. Then it is well known that $\Lambda = B_3^{-1}$. Explicitly

$$B_3\ =\ |\Sigma_n|^{-1}\begin{bmatrix} M_{33} & M_{43} \\ M_{34} & M_{44} \end{bmatrix}$$

and thus

$$\Lambda\ =\ (|\Sigma_n||B_3|)^{-1}\begin{bmatrix} M_{44} & -M_{34} \\ -M_{43} & M_{33} \end{bmatrix}$$

$$=\ |\Lambda_1|^{-1}\begin{bmatrix} M_{44} & -M_{34} \\ -M_{43} & M_{33} \end{bmatrix}\ ,$$

i.e. $\Lambda_{11} = |\Lambda_1|^{-1}M_{44}$ and $\Lambda_{22} = |\Lambda_1|^{-1}M_{33}$ which completes the proof.

For notational convenience in the proofs of the next three lemmas we define

$$\Sigma_n(t,s) = \begin{bmatrix} A & F & B & G \\ F & D & H & E \\ B & H & C & J \\ G & E & J & C \end{bmatrix}$$

$u = 2^n t - k_n(t)$, $v = 2^n s - k_n(s)$ ($k_n(t)$ being defined as in 2.2) and for integer m, $r_m = r(m2^{-n})$.

Let $\chi_n = \chi_n(t,s)$ be the indicator function for the set S_n.

<u>Lemma</u> 4.2.8: $\chi_n(t,s)\cdot|\Lambda_1|^{-1} \leq [\lambda_2\tau^2 + o(\tau^2)]^{-1}$ as $\tau = t-s \longrightarrow 0$ and the o-term is uniform in n.

Proof: We note that for $(s,t) \in S_n$ we have $|t-s| \geq 2^{-n}$. This is the only property of S_n which will be used here.

With the notation established above we have $|\Lambda_1| = AD - F^2$ where, by the proof of Lemma 4.2.5,

$$A = 1 - 2u(1-u)(1-r_1) , \qquad D = 1 - 2v(1-v)(1-r_1) ,$$

$$F = [(1-u)(1-v) + uv]r_j + (1-u)vr_{j-1} + (1-v)ur_{j+1} \text{ and } j = k_n(t)-k_n(s).$$

Now $r_1 = 1 - 2^{-2n-1}(\lambda_2-\psi)$ where $\psi = \lambda_2 + r''(\theta_1)$, $0 \leq \theta_1 \leq 2^{-n}$, and $r_m = 1 - \lambda_2 m^2\, 2^{-2n-1} + \psi_m/2$ $(m=j-1,j,j+1)$, where $\psi_m = [\lambda_2 + r''(\xi_m)]2^{-2n}\, m^2$ with $0 \leq \xi_m \leq m2^{-n}$.

Thus

$$\begin{aligned} AD &= \{1 - u(1-u)2^{-2n}(\lambda_2-\psi)\}\{1 - v(1-v)2^{-2n}(\lambda_2-\psi)\} \\ &= 1 - [u(1-u) + v(1-v)]2^{-2n}(\lambda_2-\psi) + uv(1-u)(1-v)2^{-4n}(\lambda_2-\psi)^2 \end{aligned}$$

By definition $0 \leq u, v \leq 1$ for all s,t and n. Further $\psi = \lambda_2 + r''(\theta_1) < \Psi(\theta_1)$ where Ψ is the function assumed given in the statement of the theorem. By assumption $\Psi(\tau)$ decreases as τ decreases to zero. Thus, for $|\tau| \geq 2^{-n}$,

$$|u(1-u) + v(1-v)|2^{-2n}\psi \leq \text{const. } \tau^2\,\Psi(\theta_1) \leq \text{const. } \tau^2\,\Psi(\tau)$$

and $2^{-4n}(\lambda_2-\psi)^2 \leq$ const. τ^4. Hence for $|\tau| \geq 2^{-n}$ we have

$$AD \;=\; 1 - [u(1-u) + v(1-v)]2^{-2n}\,\lambda_2 + o(\tau^3)$$

as $\tau \longrightarrow 0$ where $o(\tau^3)$ is uniform in n.

Now

$$F \;=\; 1 - [j^2 + 2j(u-v) + u + v - 2uv]2^{-2n-1}\,\lambda_2 + [(1-u)(1-v) + uv]\psi_j/2$$

$$+\ (1-u)v\ \psi_{j-1}/2 + (1-v)u\ \psi_{j+1}/2\quad .$$

For $|\tau| \geq 2^{-n}$ we have $|j+1|2^{-n} \leq 3|\tau|$ and thus

$$|1-v|u\ \psi_{j+1} \;\leq\; \text{const. } \tau^2\ \Psi(\xi_{j+1}) \;\leq\; \text{const. } \tau^2\ \Psi(3\tau)$$

and similar bounds for the other ψ_m terms follow. Hence, subject to $|\tau| \geq 2^{-n}$,

$$F^2 \;=\; 1 - [j^2 + 2j(u-v) + u + v - 2uv]2^{-2n}\,\lambda_2 + o(\tau^2)$$

where $o(\tau^2)$ is uniform in n. Therefore

$$\chi_n(t,s)\ |\Lambda_1|^{-1} \;\leq\; [(j+u-v)^2\ 2^{-2n}\,\lambda_2 + o(\tau^2)]^{-1}$$

$$=\; [\lambda_2\tau^2 + o(\tau^2)]^{-1}\quad , \qquad \text{as } \tau \longrightarrow 0\ ,$$

where $o(\tau^2)$ is uniform in n as required.

<u>Lemma</u> 4.2.9: $\quad \chi_n(t,s)\cdot M_{33} \;\leq\; K\,\tau^2\ \Psi(3\tau) + o(\tau^3)\quad$, as $\tau \longrightarrow 0$, where the o-term is uniform in n and K is a positive constant.

Proof: We have

$$(4.2.10)\qquad M_{33} \;=\; C(AD-F^2) + E(2FG-AE) - DG^2$$

and we consider the three terms separately.

From the proof of Lemma 4.2.8 we immediately obtain

$$\chi_n\cdot(AD-F^2) \;\leq\; (j+u-v)^2\ 2^{-2n}\,\lambda_2 + K\,\tau^2\ \Psi(3\tau) + o(\tau^3)$$

where K is a constant and $o(\tau^3)$ is uniform in n. Further $C = 2^{2n+1}(1-r_1) = \lambda_2 - \psi$ and hence, since $\chi_n\cdot\psi \leq \Psi(\tau)$.

$$(4.2.11) \qquad X_n \cdot C(AD-F^2) \;\leq\; (j+u-v)^2\, 2^{-2n}\, \lambda_2^2 + K\,\tau^2\, \Psi(3\tau) + o(\tau^3)\ .$$

In considering the second term of (4.2.10), we find

$$E \;=\; 2^n(2v-1)(1-r_1) \;=\; 2^{-n}(v-\tfrac{1}{2})(\lambda_2-\psi)$$

so that

$$AE \;=\; 2^{-n}(v-\tfrac{1}{2})(\lambda_2-\psi) - 2^{-3n}\, u(1-u)(v-\tfrac{1}{2})(\lambda_2-\psi)^2\ .$$

Also

$$G \;=\; 2^n[(1-u)(r_{j-1}-r_j) + u(r_j-r_{j+1})]$$

$$=\; 2^{n-1}[\lambda_2 2^{-2n}(2j+2u-1) + (2u-1)\psi_j - u\,\psi_{j+1} + (1-u)\psi_{j-1}]$$

and hence for $|\tau| > 2^{-n}$

$$2FG \;=\; 2^n[\lambda_2 2^{-2n}(2j+2u-1) + (2u-1)\psi_j + (1-u)\psi_{j-1} - \psi_{j+1}]$$

$$\cdot[1 - (j^2+2j(u-v) + u + v - 2uv)\lambda_2 2^{-2n-1} + o(\tau^2)]$$

$$=\; 2^n[\lambda_2 2^{-2n}(2j+2n-1) + (2u-1)\psi_j + (1-u)\psi_{j-1} - u\,\psi_{j+1} + o(\tau^3)].$$

The second term of (4.2.10) is therefore

$$(4.2.12) \quad E(2FG-AE) \;=\; \lambda_2^2\, 2^{-2n}(v-\tfrac{1}{2})(2j+2u-v+\tfrac{1}{2}) - 2^{-2n}\,\psi(v-\tfrac{1}{2})^2(-2\lambda_2+\psi)$$

$$+ (v-\tfrac{1}{2})(\lambda_2-\psi)[(2u-1)\psi_j + (1-u)\psi_{j-1} - u\,\psi_{j+1}] + o(\tau^3)$$

where $o(\tau^3)$ is uniform in n subject to $|\tau| \geq 2^{-n}$.

To evaluate the last term of (4.2.10) we have

$$D \;=\; 1 - 2v(1-v)(1-r_1) \;=\; 1 - v(1-v)2^{-2n}(\lambda_2-\psi)\ .$$

For G we expand r_{j-1} and r_{j+1} around $j2^{-n}$, i.e.

$$r_{j-1} \;=\; r_j - 2^{-n}\, r_j' + 2^{-2n-1}\, r''(\xi_1)\ , \qquad 2^{-n}(j-1) \leq \xi_1 \leq 2^{-n}\, j$$

and

$$r_{j+1} = r_j + 2^{-n} r_j' + 2^{-2n-1} r''(\xi_2) , \qquad 2^{-n} j \le \xi_2 \le 2^{-n}(j+1) .$$

Then

$$G = 2^n[(1-u)(r_{j-1}-r_j) + u(r_j-r_{j+1})]$$

$$= -r_j' + 2^{-n-1}[(1-u)r''(\xi_1) - ur''(\xi_2)] .$$

Now writing $r_j' = j\ 2^{-n}\ r''(\xi_3)$, $0 \le \xi_3 \le j\ 2^{-n}$ (note $r'(0) = 0$), we have

$$G = 2^{-n}\ \lambda_2(j+u-\tfrac{1}{2}) - j\ 2^{-n}\ \phi_3 + (1-u)2^{-n-1}\ \phi_1 - u\ 2^{-n-1}\ \phi_2$$

where $\phi_i = \lambda_2 + r''(\xi_i)$, $i = 1,2,3$.

Hence

$$G^2 \ge 2^{-2n}\ \lambda_2^2(j+u-\tfrac{1}{2})^2 + 2^{-2n}\ \lambda_2(j+u-\tfrac{1}{2})[-2j\phi_3 + (1-u)\phi_1 - u\phi_2]$$
$$- j2^{-2n}(1-u)\phi_1\phi_3 + j2^{-2n}\ u\phi_3\phi_2 - 2^{-2n-1}\ u(1-u)\phi_1\phi_2$$

and therefore

$$(4.2.13) \qquad -\chi_n\cdot DG^2 \le -2^{-2n}\ \lambda_2^2(j+u-\tfrac{1}{2})^2 - 2^{-2n}\ \lambda_2(j+u-\tfrac{1}{2})[-2j\phi_3 + (1-u)\phi_1 - u\phi_2]$$
$$+j2^{-2n}(1-u)\phi_1\phi_3 - j2^{-2n}\ u\phi_3\phi_2 + 2^{-2n-1}\ u(1-u)\phi_1\phi_2 + o(\tau^3)$$

as $\tau \longrightarrow 0$, where $o(\tau^3)$ is again uniform in n. The $o(\tau^3)$ comes from terms like $2^{-4n}j^2$, $2^{-4n}j^2\phi_3$ and $2^{-4n}j\phi_1$ which are all dominated by a constant multiple of τ^4 if $|\tau| \ge 2^{-n}$.

If we combine (4.2.11-13) as required in M_{33} we find that the terms <u>not</u> multiplied by ϕ_i, ψ, or ψ_i, i.e. the first term in each case, all cancel. The terms that remain in (4.2.12) include, for example, $2^{-2n}\ j\ \psi = 2^{-2n}\ j[\lambda_2 + r''(\theta_1)]$ and $u\ \psi_{j+1} = u2^{-2n-1}(j+1)^2[\lambda_2+r''(\xi_{j+1})]$ which, subject to $|\tau| \ge 2^{-n}$, are all dominated in absolute value by const. $\tau^2\ \Psi(3\tau)$. Similarly a typical remaining term in (4.2.13) is, for example, $2^{-2n}\ j^2\phi_3$ which is also dominated by const. $\tau^2\ \Psi(3\tau)$. Hence by combining these results the lemma is proved.

<u>Lemma</u> 4.2.10: Let ν_2 denote the second element of the vector Λ_2' adj. $\Lambda_1(a,a)'$ where adj. Λ_1 is the adjoint of Λ_1. Then there is a positive constant K' such that

$$\chi_n(t,s)\cdot|\nu_2| \leq K'|\tau^3| .$$

Proof: In our notation we have

$$\Lambda_2' \text{ adj. } \Lambda_1(a,a)' = \begin{bmatrix} B & H \\ G & E \end{bmatrix} \begin{bmatrix} D & -F \\ -F & A \end{bmatrix} \begin{bmatrix} a \\ a \end{bmatrix} = a \begin{bmatrix} DB-F(H+B) + AH \\ DG-F(E+G) + AE \end{bmatrix} ,$$

i.e., $\nu_2 = (D-F)G + (A-F)E$.

Using the notation of the proofs of the previous two lemmas, we obtain

$$D - F = -v(1-v)2^{-2n}(\lambda_2-\psi) + [j^2 + 2j(u-v) + u + v - 2uv]2^{-2n-1}\lambda_2$$
$$-[(1-u)(1-v) + uv]\,\psi_j/2 - (1-u)v\,\psi_{j-1}/2 - (1-v)u\,\psi_{j+1}/2$$

and

$$A - F = -u(1-u)2^{-2n}(\lambda_2-\psi) + [j^2 + 2j(u-v) + u + v - 2uv]2^{-2n-1}\lambda_2$$
$$-[(1-u)(1-v) + uv]\,\psi_j/2 - (1-u)v\,\psi_{j-1}/2 - (1-v)u\,\psi_{j+1}/2 .$$

Multiplying by G and E, respectively, we find that the largest of the terms is dominated by a constant multiple of $(|j|2^{-n})^3 \leq |2\tau|^3$, when $|\tau| \geq 2^{-n}$. Most of the terms are in fact $o(\tau^3)$. Thus we have established the existance of the constant K' and the lemma is proved.

<u>Proof of the theorem</u>:

We first note that M_{33} and M_{44} of Lemma 4.2.7 differ only in that s and t are interchanged. Thus the result of Lemma 4.2.9 is equally valid for M_{44} in place of M_{33}. Similarly the first element of Λ_2' adj. $\Lambda_1(a,a)'$, ν_1 say, differs from ν_2 only by an interchange of s and t and so Lemma 4.2.10 holds for ν_1. Hence, using also Lemmas 4.2.7 and 4.2.8 we have

$$(4.2.14)\quad \chi_n(t,s)\cdot\int_0^\infty\int_0^\infty xy\, p_{n,t,s}(a,a,x,y)dxdy \leq \frac{K\,\tau^2\,\Psi(3\tau)}{[\lambda_2\tau^2+o(\tau^2)]^{3/2}} + \frac{[0(\tau^3)]^2}{[\lambda_2\tau^2+o(\tau^2)]^{5/2}}$$

$$\leq K_1\frac{\Psi(3\tau)}{\tau} + K_2 \quad ,$$

where K_1 and K_2 are positive constants.

Now for any even function f, $\int_0^1\int_0^1 f(t-s)dsdt = 2\int_0^1(1-\tau)f(\tau)d\tau$. Thus our assumption that $\Psi(\tau)/\tau$ be integrable on (0,1) together with inequality (4.2.14) allow dominated convergence to be applied to (4.2.8) and we obtain

$$(4.2.15)\quad \mathcal{E}\{N_u^2\} = \mathcal{E}\{N_u\} + \int_0^1\int_0^1 \lim_{n\to\infty}\int_0^\infty\int_0^\infty xy\, p_{n,t,s}(a,a,x,y)dxdydsdt \quad .$$

(We recall that by Lemma 2.2.2 and monotone convergence we have

$$\mathcal{E}\{N_n\} \longrightarrow \mathcal{E}\{N_u\} \quad\text{and}\quad \mathcal{E}\{N_n^2\} \longrightarrow \mathcal{E}\{N_u^2\}.)$$

By Lemma A.2 we have

$$p_{n,t,s}(a,a,x,y) \leq \frac{1}{(2\pi)^2|\Sigma_n|^{\frac{1}{2}}}\exp[-\tfrac{1}{2}(x,y)\Lambda_3^{-1}(x,y)']$$

where Λ_3 is the covariance matrix of $Y_n'(t)$, $Y_n'(s)$. For a fixed point (s,t) with $t \neq s$ and for all sufficiently large n, $\mathrm{Cov}[Y_n'(t), Y_n'(s)] \leq \delta < 1$. Hence we may again appeal to the dominated convergence theorem to obtain

$$\mathcal{E}\{N_u^2\} = \mathcal{E}\{N_u\} + \int_0^1\int_0^1\int_0^\infty\int_0^\infty xy \lim_{n\to\infty} p_{n,t,s}(a,a,x,y)dxdydsdt$$

$$= \mathcal{E}\{N_u\} + \int_0^1\int_0^1\int_0^\infty\int_0^\infty xy\, p_{t-s}(a,a,x,y)dxdydsdt$$

$$< \infty \; .$$

This is the desired result.

We may notice that the assumption on the behavior of $\lambda_2 + r''(\tau)$ was not used until after Lemma 4.2.6. Further (4.2.8) may be obtained as an <u>inequality</u>

(by Fatou's lemma) without assuming that $F(\lambda)$ has a continuous component, i.e. if $\lambda_2 < \infty$ we have

$$\mathcal{E}\{N_n^2\} \geq \mathcal{E}\{N_n\} + \int\int_{S_n} \int_o^\infty \int_o^\infty xy\, p_{n,t,s}(a,a,x,y)dxdydsdt\ .$$

But now a further application of Fatou's lemma gives

Theorem 4.2.2: If $\lambda_2 < \infty$, then

$$(4.2.16)\quad \mathcal{E}\{N_u^2\} \geq \mathcal{E}\{N_u\} + \int_o^1 \int_o^1 \int_o^\infty \int_o^\infty xy\, p_{t-s}(a,a,x,y)dxdydsdt\ .$$

In fact the restriction $\lambda_2 < \infty$ is not really necessary (except to define p_τ) since if $\lambda_2 = \infty$, then $\mathcal{E}\{N_u^2\} = \mathcal{E}\{N_u\} = \infty$ so that (4.2.16) is satisfied.

Now we can show, by example, that the conditions of Theorem 4.2.1, which are sufficient for finiteness of $\mathcal{E}\{N_u^2\}$ are also very close to being necessary. To this end we assume that we have a covariance function $r(\tau)$ with the property that

$$(4.2.17)\qquad \lambda_2 + r''(\tau) \sim \frac{1}{|\log|\tau||}\ , \quad \text{as } \tau \longrightarrow 0\ .$$

That this is possible follows from Theorem 1 of Pitman (1960). In particular $-r''(\tau)$ is a covariance function with spectrum $\int_o^\lambda u^2 dF(u)$ and (4.2.17) will hold if we choose $F(\lambda)$ such that $1 - F(\lambda) = \dfrac{1}{2\lambda^2\log^2\lambda}$ for all sufficiently large λ. Further from Theorem 5 of Pitman (1960) this choice of $F(\lambda)$ also implies the expansion

$$(4.2.18)\qquad r(\tau) = 1 - \frac{\lambda_2}{2}\tau^2 + \frac{\tau^2}{\log|\tau|} + 0\left(\frac{\tau^2}{\log|\tau|}\right)\ , \quad \text{as } \tau \longrightarrow 0\ .$$

Now for zero level crossings, i.e. $a = 0$, some calculation shows that

$$\int_o^\infty \int_o^\infty xy\, p_\tau(0,0,x,y)dxdy \sim K\frac{[\Sigma_{33}^2 - \Sigma_{34}^2]^{\frac{1}{2}}}{[1-r^2(\tau)]^{3/2}}$$

where K is a constant (cf. equation 4.2.6). Now using the expansion (4.2.18) we find

$$\Sigma_{33}^2 - \Sigma_{34}^2 \sim \frac{\lambda_2^2\,\tau^4}{\log^2|\tau|}$$

whereas

$$1 - r^2(\tau) \sim \lambda_2 \tau^2 .$$

Hence

$$\int_0^\infty \int_0^\infty xy\, p_\tau(0,0,x,y)dxdy \sim K/(|\tau|\log|\tau|) \quad \text{as } \tau \longrightarrow 0 .$$

It follows that the right side of (4.2.16) is infinite and thus $\mathcal{E}\{N_u^2\} = \infty$. We note that $\lambda_2 + r''(\tau)$ just fails to satisfy the integrability requirement.

4.3 The Variance of the Total Number of Crossings.

In the previous section the formula for the second moment (and hence variance) of the number of upcrossings of the level a by a stationary normal process was obtained. Here we shall show how the analogous result for the total number of crossings (both up and down) may be obtained from the results of Section 2.

Let N denote the total number of crossings of the level a by the process X(t) for $0 \leq t \leq 1$. As in Section 2 we assume that $\mathcal{E}\{X(t)\} \equiv 0$ and the covariance $r(\tau)$ is such that $\lambda_2 < \infty$ and $\lambda_2 + r''(\tau) \leq \Psi(\tau)$ where $\Psi(\tau) \downarrow 0$ as $\tau \downarrow 0$ and $\Psi(\tau)/\tau$ is integrable on [0,1]. Further we assume that the (integrated) spectrum $F(\lambda)$ has a continuous component. Then we can state the following result for the second moment of N.

Theorem 4.3.1:

$$(4.3.1) \qquad \mathcal{E}\{N^2\} = \mathcal{E}\{N\} + \int_0^1 \int_0^1 \int_{-\infty}^\infty \int_{-\infty}^\infty |xy|\, p_{t-s}(a,a,x,y)dxdy ,$$

where $p_\tau(u,v,x,y)$ is the four-dimensional normal density for $X(0)$, $X(\tau)$, $X'(0)$, $X'(\tau)$.

In the zero level case we can use Lemma A.3 to obtain the somewhat more explicit result

$$(4.3.2) \quad \mathcal{E}\{N^2\} = \frac{\lambda_2^{\frac{1}{2}}}{\pi} + \frac{2}{\pi^2} \int_0^1 (1-\tau)(\Sigma_{33}^2 - \Sigma_{34}^2)^{\frac{1}{2}} [1-r^2(\tau)]^{-3/2} (1 + \Delta \tan^{-1} \Delta)d\tau ,$$

where, as in (4.2.6), the Σ_{ij} are cofactors of the matrix Σ defined by (4.2.5) and $\Delta = \Sigma_{34}(\Sigma_{33}^2-\Sigma_{34}^2)^{-\frac{1}{2}}$. This is the formula of Steinberg et al (1955).

The proof of (4.3.1) will, again, be given via several lemmas.

Lemma 4.3.1:

$$\mathcal{E}\{N^2\} = 2\lambda_2^{\frac{1}{2}} e^{-a^2/2}/\pi + 2\int_0^1\int_0^1 [\int_0^\infty\int_0^\infty + \int_{-\infty}^0\int_{-\infty}^0]xy\, p_{t-s}(a,a,x,y)dsdydsdt$$

$$-2\int_{-\infty}^a\int_a^\infty \phi[x,y;r(1)]dxdy ,$$

where $\phi(x,y;\rho)$ is the standardized bivariate normal density function with correlation coefficient ρ.

Proof: We first note that the assumption $\lambda_2 < \infty$ is sufficient to ensure that $N < \infty$ with probability one and thus, by Lemma 2.1.2, $N = N_u + N_d$, where N_d is the number of downcrossings on [0,1]. Thus

$$\mathcal{E}\{N^2\} = \mathcal{E}\{N_u^2\} + \mathcal{E}\{N_d^2\} + 2\mathcal{E}\{N_uN_d\} .$$

By Lemma 2.1.4 we obtain

$$\mathcal{E}(N_u-N_d)^2 = P\{X(0) < a < X(1) \text{ \underline{or} } X(1) < a < X(0)\}$$

$$= 2\int_{-\infty}^a\int_a^\infty \phi[x,y;r(1)]dxdy .$$

But also

$$\mathcal{E}(N_u-N_d)^2 = \mathcal{E}\{N_u^2\} + \mathcal{E}\{N_d^2\} - 2\mathcal{E}\{N_uN_d\}$$

Hence

$$\mathcal{E}\{N^2\} = 2\mathcal{E}\{N_u^2\} + 2\mathcal{E}\{N_d^2\} - 2\int_{-\infty}^a\int_a^\infty \phi[x,y;r(1)]dxdy .$$

(Note that $\mathrm{Cov}[N_u,N_d]$ may also be obtained at once from this derivation).

Now X(t) has a downcrossing of a at t_o if and only if $-X(t)$ has an upcrossing of -a at t_o. Thus by Theorem 4.2.1

$$\mathcal{E}\{N_d^2\} = \mathcal{E}\{N_d\} + \int_o^1\int_o^1\int_o^\infty\int_o^\infty xy\, p_{t-s}(-a,-a,x,y)\,dxdydsdt$$

$$= \mathcal{E}\{N_u\} + \int_o^1\int_o^1\int_{-\infty}^o\int_{-\infty}^o xy\, p_{t-s}(a,a,x,y)\,dxdydsdt$$

and since $\mathcal{E}\{N\} = 2\mathcal{E}\{N_u\} = 2\mathcal{E}\{N_d\}$ we obtain the lemma.

Next we need the following result.

Lemma 4.3.2: Let the covariance matrix Σ be partitioned into 2 x 2 matrices as

$$\Sigma = \begin{bmatrix} \Sigma_1 & \Sigma_2 \\ \Sigma_2' & \Sigma_3 \end{bmatrix}.$$

Then

$$\Sigma_2' \Sigma_1^{-1} \Sigma_2 = \frac{(r')^2}{1-r^2} \begin{bmatrix} 1 & r \\ r & 1 \end{bmatrix}$$

$$\Sigma_2' \Sigma_1^{-1} \begin{bmatrix} a \\ a \end{bmatrix} = \frac{r'}{1+r} \begin{bmatrix} a \\ -a \end{bmatrix}$$

where the argument of r and r' has been suppressed.

Proof: The proof is by straight-forward calculation once the identification

$$\Sigma_1 = \begin{bmatrix} 1 & r \\ r & 1 \end{bmatrix} \qquad \Sigma_2 = \begin{bmatrix} 0 & -r' \\ r' & 0 \end{bmatrix}$$

is made.

Now by considering the xy integration in (4.3.1) in each of the four quadrants and then using Lemma 4.3.1, we see that the theorem will follow if we can show the next lemma.

Lemma 4.3.3:

$$\lambda_2^{\frac{1}{2}} e^{-a^2/2}/\pi + \int_0^1\int_0^1 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy\, p_{t-s}(a,a,x,y)dxdydsdt$$

$$= 2 \int_{-\infty}^{a} \int_{a}^{\infty} \phi(x,y;\rho)dxdy ,$$

where $\rho = r(1)$.

Proof: As in the proof of Lemma 4.2.7, we write

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, p_\tau(a,a,x,y)dxdy = \frac{\exp[-\underline{a}'\ \Sigma_1^{-1}\ \underline{a}]}{2\pi|\Sigma_1|^{\frac{1}{2}}} \cdot \frac{|R_3|^{\frac{1}{2}}}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, \exp[-(\underline{x}-\underline{u})'R_3(\underline{x}-\underline{u})]d\underline{x},$$

where $\underline{a}' = (a,a)$, $\underline{x}' = (x,y)$, $R_3^{-1} = \Sigma_3 - \Sigma_2'\ \Sigma_1^{-1}\ \Sigma_2$ and $\underline{u} = \Sigma_2'\ \Sigma_1^{-1}\ \underline{a}$.

The right-hand side may easily be evaluated by considering it as a bivariate product moment to obtain

$$\frac{\exp[-\underline{a}'\ \Sigma_1^{-1}\ \underline{a}]}{2\pi|\Sigma_1|^{\frac{1}{2}}} \{[\Sigma_3-\Sigma_2'\ \Sigma_1^{-1}\ \Sigma_2]_{12} + [\Sigma_2'\ \Sigma_1^{-1}\ \underline{a}]_1 \cdot [\Sigma_2'\ \Sigma_1^{-1}\ \underline{a}]_2\} ,$$

where subscripts on square brackets indicate the elements of the matrix or vector to be taken. Thus by Lemma 4.3.2 we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, p_\tau(a,a,x,y)dxdy$$

(4.3.3)

$$= \frac{\exp[-a^2/(1+r)]}{2\pi(1-r^2)^{3/2}} \{-r''(1-r^2) - r(r')^2 - a^2(r')^2[\tfrac{1-r}{1+r}]\}$$

where $r = r(\tau)$, $r' = r'(\tau)$, $r'' = r''(\tau)$ and $\tau = t-s$.

Now for any function $f(x)$ whose derivative $f'(x)$ is even and integrable we have

$$\int_0^1\int_0^1 f'(t-s)dsdt = 2\int_0^1 (1-\tau)\, f'(\tau)d\tau$$

and integrating by parts gives

$$\int_0^1\int_0^1 f'(t-s)dsdt = 2 \lim_{\tau\to 0} f(\tau) + 2\int_0^1 f(\tau)d\tau .$$

Noting that the right side of (4.3.3) is just $-\frac{d}{d\tau}[\frac{r' \exp[-a^2/(1+r)]}{2\pi(1-r^2)^{\frac{1}{2}}}]$

we have

$$\int_0^1\int_0^1\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, p_\tau(a,a,x,y)dxdydsdt$$

$$= \lim_{\tau\to 0} \frac{r' \exp[-a^2/(1+r)]}{2\pi(1-r^2)^{\frac{1}{2}}} - \int_0^1 \frac{r'(\tau) \exp[-a^2/(1+r(\tau))]}{\pi[1-r^2(\tau)]^{\frac{1}{2}}} d\tau .$$

Using the expansion $r(\tau) = 1 - \lambda_2\tau^2/2 + o(\tau^2)$ it is easy to show that the limit on the right is just $-\lambda_2^{\frac{1}{2}} e^{-a^2/2}/\pi$ so that the proof of the lemma, and hence the theorem, will be complete once we obtain

$$-\int_0^1 \frac{r'(\tau) \exp[-a^2/(1+r(\tau))]}{\pi[1-r^2(\tau)]^{\frac{1}{2}}} d\tau = 2\int_{-\infty}^{a}\int_{a}^{\infty} \phi(x,y;\rho)dxdy .$$

Working with the left-hand side we make the substitution $r(\tau) = s$ which transforms this into

$$\int_\rho^1 \frac{\exp[-a^2/(1+s)]}{\pi(1-s^2)^{\frac{1}{2}}} .$$

Now consider the right-hand side. Cramér (1963) has given the useful relation

$$\int_a^{\infty}\int_b^{\infty} \phi(x,y;\rho)dxdy = \int_a^{\infty} \phi(x)dx \cdot \int_b^{\infty} \phi(y)dy + \int_0^{\rho} \phi(a,b;s)ds$$

which is true for any real a,b and any ρ satisfying $|\rho| < 1$.

Hence

$$\int_{-\infty}^{a}\int_{a}^{\infty}\phi(x,y;\rho)dxdy = \int_{-a}^{\infty}\int_{a}^{\infty}\phi(x,y;-\rho)dxdy$$

$$= \int_{-a}^{\infty}\phi(x)dx\cdot\int_{a}^{\infty}\phi(y)dy + \int_{\rho}^{o}\frac{\exp[-a^2/(1+s)]}{2\pi(1-s^2)^{\frac{1}{2}}}ds$$

Therefore it remains to show that

$$\int_{o}^{1}\frac{\exp[-a^2/(1+s)]}{\pi(1-s^2)^{\frac{1}{2}}}ds = 2\int_{-a}^{\infty}\phi(x)dx\cdot\int_{a}^{\infty}\phi(y)dy \quad . \tag{4.3.4}$$

If we write the right-hand side as a double integral and then change to polar coordinates we find

$$\int_{-a}^{\infty}\int_{a}^{\infty}\phi(x)\ \phi(y)dxdy = \int_{o}^{\pi/2}\int_{a/\sin\theta}^{\infty}\frac{e^{-\frac{1}{2}R^2}}{2\pi}RdRd\theta + \int_{\pi/2}^{3\pi/4}\int_{a/\sin\theta}^{a/\cos\theta}\frac{e^{-\frac{1}{2}R^2}}{2\pi}RdRd\theta \ .$$

Carrying out the integration over R and then making the substitutions $2\sin^2\theta = 1+s$ in one place and $2\cos^2\theta = 1+s$ in another we are finally led to the left-hand side of (4.3.4).

Thus the lemma and hence the theorem are proved.

4.4 Covariance of the Number of Upcrossings in Disjoint Intervals.

Let, as before, N_u denote the number of upcrossings of the level a by X(t) for $0 \le t \le 1$ and let M_u denote the same quantity for $T - 1 \le t \le T$. Then the moment $\mathcal{E}\{N_u\cdot M_u\}$ is of interest and we shall now show how it may be derived by methods completely analogous to those of Section 2. If $1 < T < 2$ let K_u be the number of upcrossings by X(t) for $T - 1 \le t \le 1$, i.e. on the overlapping interval. We now state the main result using the notation of Section 2.

<u>Theorem</u> 4.4.1: We assume that $F(\lambda)$ has a continuous component and that λ_2 is finite.

(i) If $1 < T \le 2$ and there is a function $\Psi(\tau)$ as in Theorem 4.2.1, then

$$\mathcal{E}\{N_u\cdot M_u\} = \mathcal{E}\{K_u\} + \int_{T-1}^{T}\int_{o}^{1}\int_{o}^{\infty}\int_{o}^{\infty} xy\ p_{t-s}(a,a,x,y)dxdydsdt \ . \tag{4.4.1}$$

(ii) If $T > 2$, then

$$(4.4.2) \qquad \mathcal{E}\{N_u \cdot M_u\} = \int_{T-1}^{T} \int_{o}^{1} \int_{o}^{\infty} \int_{o}^{\infty} xy\, p_{t-s}(a,a,x,y)\,dxdydsdt \ .$$

In both cases the moments are finite.

Proof: The proof will be only sketched since it is similar to that of Section 2.

We first observe that we need only prove (4.4.2) since (4.4.1) follows from (4.4.2) together with the variance (4.2.3). To see this suppose $1 < T \leq 2$ and let A, K_u and B denote respectively the number of upcrossings for t in (0, T-1), (T-1, 1), and (1,T). Then $N_u = A + K_u$ and $M_u = K_u + B$. Hence

$$\mathcal{E}\{N_u \cdot M_u\} = \mathcal{E}\{(A+K_u)(K_u+B)\} = \mathcal{E}\{K_u^2\} + \mathcal{E}\{A\ K_u\} + \mathcal{E}\{K_u\ B\} + \mathcal{E}\{A\ B\}$$

$$= \mathcal{E}\{K_u\} + \left[\int_{T-1}^{1}\int_{T-1}^{1} + \int_{T-1}^{1}\int_{o}^{T-1} + \int_{1}^{T}\int_{T-1}^{1} + \int_{1}^{T}\int_{o}^{T-1}\right] \int_{o}^{\infty}\int_{o}^{\infty} xy\, p_{t-s}(a,a,x,y)$$

dxdydsdt

$$= \mathcal{E}\{K_u\} + \int_{T-1}^{T}\int_{o}^{1}\int_{o}^{\infty}\int_{o}^{\infty} xy\, p_{t-s}(a,a,x,y)\,dxdydsdt \ ,$$

where of course we are using the fact that (4.4.2) holds for <u>any</u> two disjoint intervals with obvious modifications. Hence we only prove (4.4.2).

Let N_n and M_n denote the number of upcrossings by $Y_n(t)$ in (0,1), (T-1, T), respectively. ($Y_n(t)$ is defined on (T-1, T) in the same manner as on (0,1), i.e. (T-1, T) is divided into 2^n+1 equal intervals and then $Y_n(t) = X(t)$ at the end points of the intervals. Between such points, $Y_n(t)$ is linear.)

Exactly as in Lemma 4.2.1 we have (with probability one)

$$M_n = \lim_{m \to \infty} \int_{T-1}^{T} \delta_n[Y_n(t)-a]\sigma[Y_n'(t)]dt$$

and hence Lemma 4.2.2 becomes

$$(4.4.3) \qquad N_n M_n = \lim_{m \to \infty} \int_{T-1}^{T}\int_{o}^{1} \delta_n[Y_n(t)-a]\delta_n[Y_n(s)-a]\sigma[Y_n'(t)]\sigma[Y_n'(s)]dsdt$$

where the (s,t) integration may be taken as such since for disjoint intervals s and t are in "separated intervals" as soon as n satisfies $2^{-n+1} < T - 2$. Further the integral in (4.4.3) is again dominated by 2^{2n} so that the analog of Lemma 4.2.3 is

$$\mathcal{E}\{N_n \cdot M_n\} = \lim_{m\to\infty} \int_{T-1}^{T} \int_{0}^{1} \mathcal{E}\{\delta_m[Y_n(t)-a]\delta_m[Y_n(s)-a]\sigma[Y_n'(t)]\sigma[Y_n'(s)]\}dsdt.$$

Lemma 4.2.5 is still valid for $0 \leq s \leq 1 \leq T - 1 \leq t \leq T$ and Lemma 4.2.4 can be strengthened somewhat to give $|\Sigma_n(s,t)| \geq C > 0$ which holds uniformly for $0 \leq s \leq 1 < T - 1 \leq t \leq T$.

Therefore we can obtain

$$\mathcal{E}\{N_n \cdot M_n\} = \int_{T-1}^{T} \int_{0}^{1} \int_{0}^{\infty} \int_{0}^{\infty} xy\, p_{n,t,s}(a,a,x,y)dxdydsdt$$

as in Lemma 4.2.6.

Now in our present case since s and t are in disjoint intervals we need not be concerned about the "singularity" of $p_{n,t,s}$ at $s = t$ and we can proceed to dominate $p_{n,t,s}(a,a,x,y)$ by a function independent of n and integrable as required so that, using dominated convergence,

$$\mathcal{E}\{N_u \cdot M_u\} = \lim_{n\to\infty} \mathcal{E}\{N_n \cdot M_n\} = \lim_{n\to\infty} \int_{T-1}^{T} \int_{0}^{1} \int_{0}^{\infty} \int_{0}^{\infty} xy\, p_{n,t,s}(a,a,x,y)dxdydsdt$$

$$= \int_{T-1}^{T} \int_{0}^{1} \int_{0}^{\infty} \int_{0}^{\infty} xy\, p_{t-s}(a,a,x,y)dxdydsdt .$$

Clearly the lower bound analogous to that given in Theorem 4.2.2 is also valid.

4.5 Asmptotic Results for Var{N(T)}.

We wish now to consider the limiting behavior of the variance of the number of crossings of the fixed level a by X(t) for $0 \leq t \leq T$ as $T \to \infty$. For this purpose we extend our notation slightly and write N(T) for the number of such crossings to show the dependence on T.

We assume throughout this section that the conditions of Theorem 4.2.1 hold and we write

(4.5.1) $$g(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |xy|\, p_\tau(a,a,x,y)dxdy - \lambda_2\, e^{-a^2}/\pi^2 .$$

Noting that $\int_0^T (1-\frac{\tau}{T})d\tau = T/2$ and using Theorem 4.3.1 and Equation (3.1.1) we have

(4.5.2) $$\mathrm{Var}\{N(T)\} = (\lambda_2^{\frac{1}{2}}T/\pi)\exp(-a^2/2) + 2T\int_0^T (1-\frac{\tau}{T})\, g(\tau)d\tau .$$

Now if $\lim_{T\to\infty} \int_0^T (1-\frac{\tau}{T})\, g(\tau)d\tau$ exists (finitely) then we have $\mathrm{Var}\{N(T)\} \sim c\,T$ where c is a constant. If $g(\tau)$ is (absolutely) integrable on $(0,\infty)$ then by dominated convergence

$$\int_0^T (1-\frac{\tau}{T})\, g(\tau)d\tau \to \int_0^\infty g(\tau)d\tau .$$

However, integrability of $g(\tau)$ is not <u>necessary</u> for convergence; e.g.

$$\int_0^T (1-\frac{\tau}{T}) \cos\tau d\tau = \frac{1-\cos\tau}{T} \to 0 .$$

Nevertheless integrability of $g(\tau)$ seems to be the most convenient condition, and we will show that it holds under rather mild conditions on $r(\tau)$. First however we give a result for a very special case.

<u>Theorem</u> 4.5.1: If $r(\tau) = 0$ for $\tau \geq \tau_o$, then for $T \geq \tau_o$ we have

(4.5.3) $$\mathrm{Var}\{N(T)\} = T\{\lambda_2^{\frac{1}{2}} \frac{\exp(-a^2/2)}{\pi} + 2\int_0^{\tau_o} g(\tau)d\tau\} - 2\int_0^{\tau_o} \tau g(\tau)d\tau ,$$

i.e. the variance is exactly linear in T, for all sufficiently large T.

Proof: By our assumption, for $\tau \geq \tau_o$, $r(\tau) = r'(\tau) = r''(\tau) = 0$. Thus for τ in this range $p_\tau(a,a,x,y) = \frac{1}{(2\pi)^2\lambda_2} \exp[-\frac{1}{2}(2a^2 + \frac{x^2}{\lambda_2} + \frac{y^2}{\lambda_2})]$ and

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |xy| \; p_\tau(a,a,x,y)dxdy \;=\; \frac{\lambda_2}{\pi^2} e^{-a^2}$$

i.e. $g(\tau) = 0$. Hence the lemma follows.

We might expect similar results if $r(\tau) \longrightarrow 0$ as $\tau \longrightarrow \infty$ or if, say, $r(\tau)$ is integrable on $(0,\infty)$. In order to discuss such questions, some preliminary lemmas will be needed.

<u>Lemma</u> 4.5.1: For $\tau > 0$ we have

(4.5.4) $$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |xy| \; p_\tau(a,a,x,y)dxdy \;=\; \frac{1}{2\pi} \exp[-a^2/(1+r)] \; A(\tau) \; I(b,h) \;,$$

where

$$A(\tau) \;=\; \frac{\lambda_2(1-r^2) - (r')^2}{(1-r^2)^{3/2}} \quad ,$$

$$b = b(\tau) \;=\; \frac{r''(1-r^2) + r(r')^2}{\lambda_2(1-r^2) - (r')^2} \quad , \qquad |b| < 1 \;,$$

(4.5.6)

$$h = h(\tau) \;=\; \frac{a\, r'}{1+r} \left[\frac{1-r^2}{\lambda_2(1-r^2) - (r')^2}\right]^{\frac{1}{2}} \quad ,$$

$$I(b,h) \;=\; \frac{1}{2\pi(1-b^2)^{\frac{1}{2}}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |(x-h)(y+h)| \; \exp\{-\frac{1}{2(1-b^2)} (x^2+2bxy+y^2)\} dxdy,$$

and $r = r(\tau)$, $r' = r'(\tau)$, $r'' = r''(\tau)$.

Proof: See Rice (1958) or Volkonski and Rozanov (1961). The proof also follows easily from the first equation in the proof of Lemma 4.2.7.

Lemma 4.5.2: Suppose $F(\lambda)$ is absolutely continuous with density $f(\lambda)$. If $f(\lambda)$ is bounded and $O(\lambda^{-2})$ as $\lambda \longrightarrow \infty$ then $r(\tau)$, $r'(\tau)$ and $r''(\tau)$ are all square integrable on $(0,\infty)$ and all tend to zero as $\tau \longrightarrow \infty$.

Proof: By our assumptions $[\lambda^2 f(\lambda)]^2 \leq$ const. $\lambda^2 f(\lambda)$. Hence since λ_2 is finite $\lambda^2 f(\lambda)$, $\lambda f(\lambda)$ and $f(\lambda)$, the Fourier transforms of $r''(\tau)$, $r'(\tau)$ and $r(\tau)$, respectively, are all square integrable on $(0,\infty)$. Thus by Parseval's theorem $r''(\tau)$, $r'(\tau)$ and $r(\tau)$ are all square integrable. Further by the Riemann-Lebesgue lemma they each tend to zero as required.

The behavior of $I(b,h)$ as $b,h \longrightarrow 0$ is now considered.

Lemma 4.5.3: As $b,h \longrightarrow 0$ we have

$$I(b,h) = \frac{2}{\pi} + O(h^2) + O(b^2) . \tag{4.5.7}$$

Proof: Let $\phi(x,y;\rho)$ denote the bivariate normal density with zero means, unit variances and correlation coefficient ρ. Then define

$$J(h,k;\rho) = \int_h^\infty \int_k^\infty (x-k)(y-h)\ \phi(x,y;\rho)dxdy . \tag{4.5.8}$$

From Cramér (1946) we have the expansion

$$\phi(x,y;\rho) = \sum_{j=1}^{\infty} \frac{\rho^{j-1}}{(j-1)!} \Phi^{(j)}(x)\ \Phi^{(j)}(y) ,$$

where $\Phi^{(j)}$ denotes the j-th derivative of the normal distribution function. Extensive use will be made of this expansion in the next chapter and we defer discussion of such questions as convergence, interchange of summation and integration, etc. to 5.3. If we substitute the expansion into (4.5.8) and integrate each term by parts we obtain

$$\begin{aligned} J(h,k;\rho) = &\{\phi(h)-h[1-\Phi(h)]\}\{\phi(k)-k[1-\Phi(k)]\} + \rho[1-\Phi(h)][1-\Phi(k)] \\ &+ \rho^2\phi(h)\phi(k) + \sum_{j=2}^{\infty} \frac{\rho^{j+1}}{(j+1)!} \Phi^{(j)}(h)\ \Phi^{(j)}(k) . \end{aligned}$$

From the proof that the sum (5.3.4) is convergent it follows that the infinite sum term in the above is $o(\rho^2)$ as $\rho \to 0$ uniformly in h and k and using the expansions $\phi(x) = (2\pi)^{-\frac{1}{2}} (1-\frac{1}{2}x^2) + o(x^2)$, $\Phi(x) = \frac{1}{2} + (2\pi)^{-\frac{1}{2}}x + o(x^2)$, as $x \to 0$, we find

$$(4.5.9) \quad J(h,k;\rho) = \frac{1}{2\pi} - \frac{h+k}{2(2\pi)^{\frac{1}{2}}} + \frac{hk}{4} + \frac{h^2+k^2}{4\pi} + \frac{\rho}{4} - \frac{\rho(h+k)}{2(2\pi)^{\frac{1}{2}}} + \frac{\rho hk}{2\pi} + \frac{\rho^2}{2\pi} + o(h^2)$$

$$+ o(k^2) + o(\rho^2) ,$$

as ρ, h, $k \to 0$.

But it follows from the definitions (4.5.6) and (4.5.8) that

$$I(b,h) = J(h,-h;-b) + J(h,h;b) + J(-h,-h;b) + J(-h,h;-b) .$$

Hence from (4.5.9) we obtain

$$I(b,h) = \frac{2}{\pi} + 0(h^2) + 0(b^2) ,$$

the desired result.

Some useful theorems can now be obtained.

<u>Theorem</u> 4.5.12: If $r(\tau)$ is integrable on $(0,\infty)$, λ_2 is finite and the spectral density satisfies

$$f(\lambda) = 0(1/\lambda^2) , \quad \text{as } \lambda \to \infty,$$

then

$$\mathrm{Var}\{N(T)\} \sim c\,T \quad , \quad \text{as } T \to \infty,$$

where

$$c = \frac{\lambda_2^{\frac{1}{2}}}{\pi} e^{-a^2/2} + 2 \int_0^\infty g(\tau)d\tau .$$

Proof: We first note that since $r(\tau)$ is integrable, the spectral density $f(\lambda)$ exists and since $f(\lambda) = \frac{2}{\pi} \int_0^\infty \cos\lambda\tau\, r(\tau)d\tau$, $f(\lambda)$ is bounded by $\frac{2}{\pi} \int_0^\infty |r(\tau)|d\tau$. Thus by Lemma 4.5.2 $r(\tau)$, $r'(\tau)$ and $r''(\tau)$ all tend to zero as $\tau \to \infty$ and are all square integrable.

Now, as $\tau \longrightarrow \infty$, we have

$$A(\tau) = \lambda_2 + \frac{\lambda_2}{2} r^2 - r'^2 + o(r^2) \quad ,$$

$$\exp[-a^2 r/(1+r)] = 1 - a^2 r + a^2(1+a^2/2)\, r^2 + o(r^2) \quad ,$$

$$b(\tau) = \frac{r''}{\lambda_2} + o(r^2) + o(r'^2) \quad ,$$

and

$$h(\tau) = \frac{ar'}{\lambda_2^{\frac{1}{2}}} - \frac{arr'}{\lambda_2^{\frac{1}{2}}} + o(r^2) + o(r'^2) \quad .$$

Thus from Lemma 4.5.3

$$I(b,h) = \frac{2}{\pi} + O(r''^2) + O(r^2) + O(r'^2) \ .$$

Now by (4.5.4) $g(\tau) = e^{-a^2}\{(2\pi)^{-1} \exp[-a^2 r/(1+r)]\, A(\tau)\, I(b,h) - \lambda_2/\pi^2\}$ and hence

$$g(\tau) = -\frac{a^2}{\pi^2}\, r(\tau)\, e^{-a^2} + O(r^2) + O(r'^2) + O(r''^2) \ . \tag{4.5.10}$$

But under our assumptions $r(\tau)$, $r^2(\tau)$, $[r'(\tau)]^2$ and $[r''(\tau)]^2$ are all integrable on $(0,\infty)$. Thus $g(\tau)$ is integrable and, as noted earlier, the result follows from dominated convergence.

With further assumptions we may refine these results. The assumptions are perhaps somewhat less elegant, but are easy to check and hold in many situations of practical interest.

<u>Theorem</u> 4.5.3: If $\tau r(\tau)$, $\tau r^2(\tau)$, $\tau[r'(\tau)]^2$ and $\tau[r''(\tau)]^2$ are all integrable on $(0,\infty)$, then

$$\mathrm{Var}\{N(T)\} = d + c\,T + o(1) \ , \quad \text{as } T \longrightarrow \infty \ ,$$

where

$$c = \frac{\lambda_2^{\frac{1}{2}}}{\pi} e^{-a^2/2} + 2\int_0^\infty g(\tau)d\tau \quad ,$$

and

$$d = -2 \int_0^\infty \tau\, g(\tau)d\tau .$$

Proof: We have

$$\mathrm{Var}\{N(T)\} - d - c\,T = -2 \int_T^\infty g(\tau)d\tau + \int_T^\infty \tau\, g(\tau)d\tau .$$

By our assumptions and equation (4.5.10) we have that both $g(\tau)$ and $\tau g(\tau)$ are integrable. Hence the proof follows.

The numerical results which follow show that this asymptotic linearity is obtained quite quickly in several cases.

4.6 Numerical Results.

(i) $r(\tau) = (1 + |\tau|)\, e^{-|\tau|}$.

For this case the spectral density is

$$f(\lambda) = \frac{4}{\pi(1+\lambda^2)^2} \quad , \quad \lambda \geq 0 .$$

Further $r'(\tau) = -\tau\, e^{-|\tau|}$, $r''(\tau) = (|\tau|-1)\, e^{-|\tau|}$ and $\lambda_2 = -r''(0) = 1$. We note that the assumptions of Theorem 4.5.3 are clearly satisfied. Using this covariance function Var{N(T)} was numerically evaluated for the number of zero crossings using an obvious modification of Equation (4.3.2). The calculated values of Var{N(T)} versus T together with the linear approximation given by Theorem 4.5.3 are plotted in Figure 4.6.1.

(ii) $r(\tau) = \cos\lambda_o\tau\, e^{-\frac{1}{2}\sigma^2\tau^2}$.

If σ/λ_o is small (we shall use $\sigma/\lambda_o = .15$) this corresponds to a very good approximation to the spectral density

$$f(\lambda) = \frac{1}{(2\pi)^{\frac{1}{2}}}\, e^{-\frac{1}{2}\frac{(\lambda-\lambda_o)^2}{\sigma^2}} \quad , \quad \lambda \geq 0 ,$$

i.e. the process has a "Gaussian spectrum centered at the frequency λ_o."

In this case

$$r'(\tau) = (-\lambda_o \sin\lambda_o\tau - \sigma^2\tau \cos\lambda_o\tau)\ e^{-\frac{1}{2}\sigma^2\tau^2}$$

$$r''(\tau) = [-(\lambda_o^2 + \sigma^2 - \sigma^4\tau^2)\cos\lambda_o\tau + 2\lambda_o\sigma^2\tau \sin\lambda_o\tau]\ e^{-\frac{1}{2}\sigma^2\tau^2}$$

$$\lambda_2 = -r''(0) = \lambda_o^2 + \sigma^2 \quad .$$

Again the assumptions of Theorem 4.5.3 are satisfied. The variance of the number of zero crossings with $\lambda_o = 2\pi$, $\sigma = (.15)2\pi$ is plotted in Figure 4.6.2 along with the linear approximation.

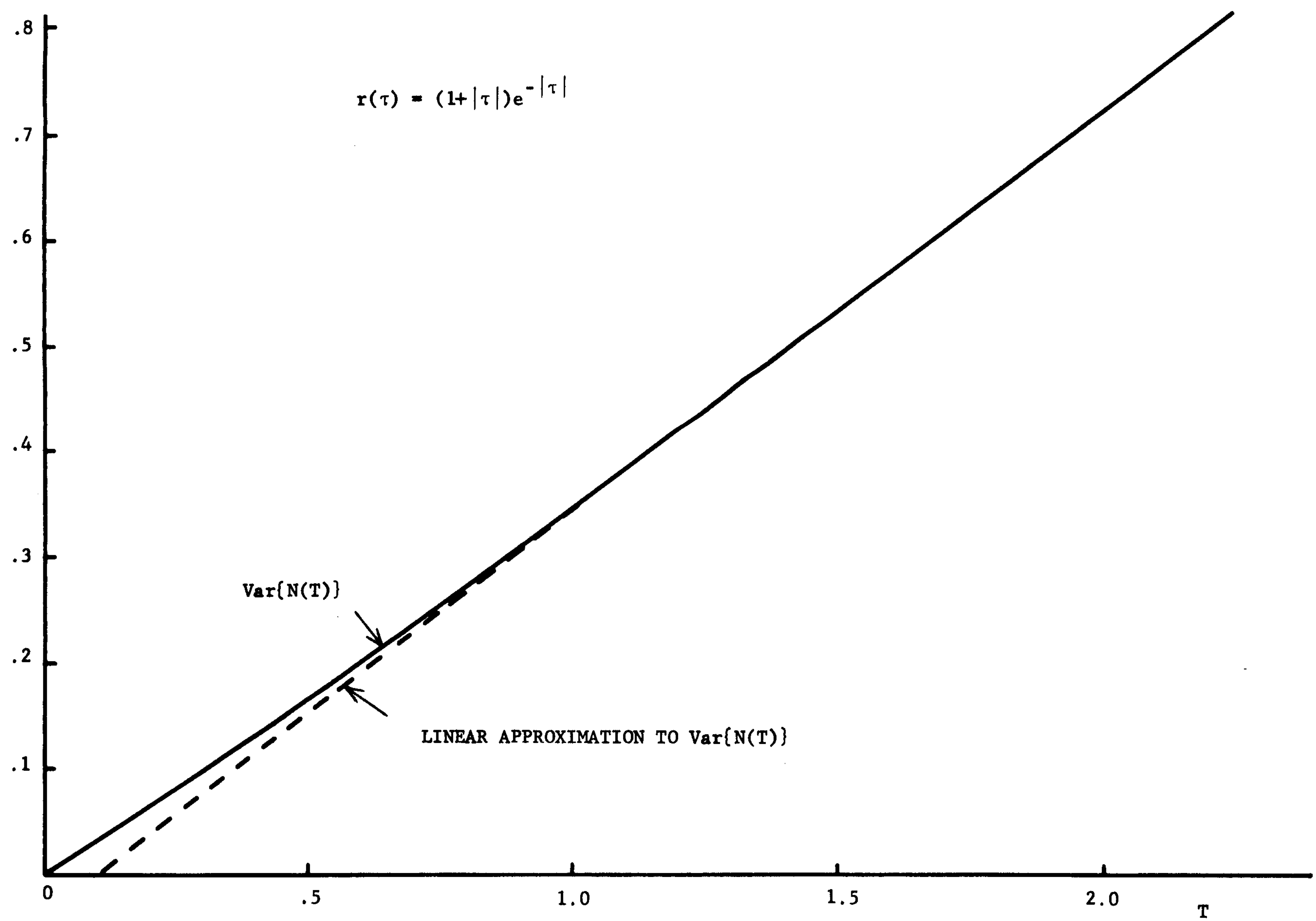
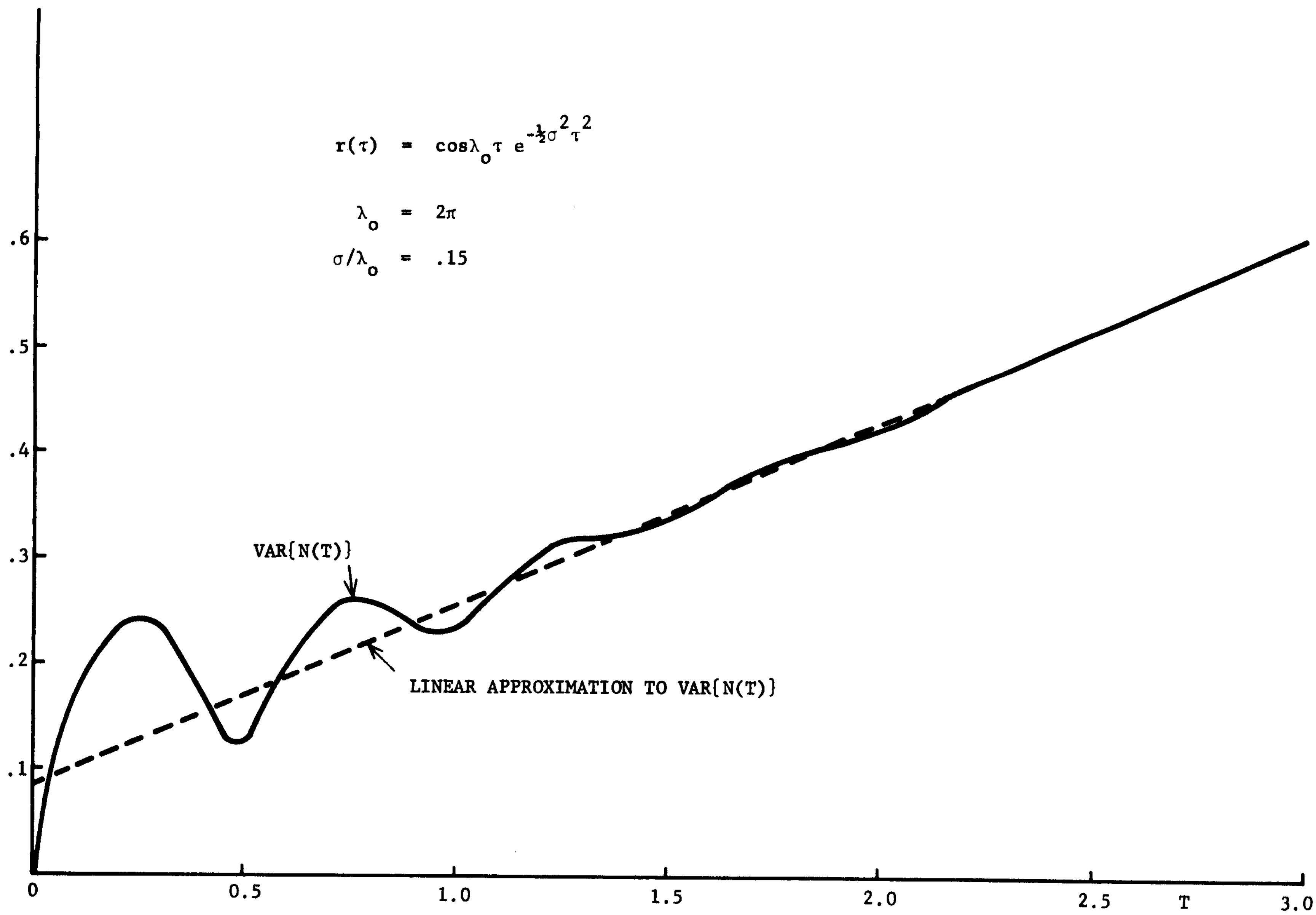

.8
.7
.6
.5
.4
.3
.2
.1
0
.5
1.0
1.5
2.0
T
r(τ) = (1+|τ|)e^{-|τ|}
Var{N(T)}
LINEAR APPROXIMATION TO Var{N(T)}

Figure 4.6.1



$r(\tau) = \cos\lambda_o\tau \, e^{-\frac{1}{2}\sigma^2\tau^2}$
$\lambda_o = 2\pi$
$\sigma/\lambda_o = .15$
VAR{N(T)}
LINEAR APPROXIMATION TO VAR{N(T)}
.6
.5
.4
.3
.2
.1
0
0.5
1.0
1.5
2.0
2.5
T
3.0

Figure 4.6.2

CHAPTER V

CERTAIN FUNCTIONALS OF NORMAL PROCESSES

5.1 Introduction.

As noted in Chapter I, when considering the performance quality or reliability of complex physical systems, it is sometimes convenient to define certain performance "indices" or "measures" based on the characteristics of a stochastic process associated with the system.

Suppose that we have such a stochastic process $\{X(t): t \in [0,T]\}$ and that for good performance $X(t)$ should never become "too large." Specifically, suppose there is a known function $a(t)$ such that for good performance $X(t)$ should always be kept less than $a(t)$. Let h be a function which is zero for negative arguments and strictly positive for positive arguments. Define the functional

$$Z = \frac{1}{T}\int_0^T h[X(t)-a(t)]\,dt \,. \tag{5.1.1}$$

(We assume that the behavior of $h(t)$, $a(t)$ and $X(t)$ is such that Z is defined.)

Now if with probability one $X(t)$ has continuous sample functions, then the event $\{Z = 0\}$ is equivalent to the event $\{X(t) \le a(t),\ t \in [0,T]\}$ and hence

$$P\{Z = 0\} = P\{X(t) \le a(t),\ t \in [0,T]\} \,,$$

and, for example, Chebyshev bounds on $P\{Z = 0\}$ (using the mean and variance of Z) give bounds on $P\{X(t) \le a(t),\ t \in [0,T]\}$. This latter quantity represents, of course, the reliability or probability of a successful mission from this point of view.

We wish to investigate forms of the function h which lead to tractable Chebyshev bounds, i.e., tractable formulae for $\mathcal{E}\{Z\}$ and $\mathrm{Var}\{Z\}$. One particularly amenable choice for h is

$$h(x) = h_n(x) = \begin{cases} x^n, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 . \end{cases}$$

We then write Z_n for Z with the corresponding h_n as integrand in (5.1.1). Note that Z_o is the proportion of time which the process spends above the curve $a(t)$ on the interval $[0,T]$ and $T \cdot Z_1$ is the area which the process cuts off above the curve on $[0,T]$.

The Z_n's may be called "exceedance measures" for the process as they describe, in various ways, excursions of $X(t)$ above $a(t)$; for example Z_o takes no account of the size of such excursions whereas the remaining Z_n's do. The first two Z_n's, Z_o and Z_1, have been considered previously by Leadbetter (1963) and Cryer (1963) for a normal, stationary process. The generalization of these results to include all Z_n is the purpose of this chapter.

5.2 The Mean and Variance of Z_n.

We assume in the remainder of this chapter that $\{X(t): t \in [0,T]\}$ is a stationary normal process with zero mean and covariance function $r(\tau)$, assumed such that, with probability one, the sample functions are everywhere continuous. Sufficient conditions for this latter property are given, for example, in Belayev (1961). With no loss of generality we take $r(0) = 1$, and we assume $|r(\tau)| < 1$ a.e.

Let $a(t)$ be a continuous function defined on $[0,T]$ and define

$$Y_n(t) = h_n[X(t)-a(t)]$$

(No confusion with the $Y_n(t)$ of previous chapters should result). We will sometimes use $a(t) = a_t$ for notational simplicity. Then Z_n can be written

$$Z_n = \frac{1}{T} \int_o^T Y_n(t)\, dt . \tag{5.2.1}$$

The mean value of Z_n is (by Fubini's theorem)

$$\mathcal{E}\{Z_n\} = \frac{1}{T}\,\mathcal{E}\{\int_o^T Y_n(t)dt\}$$

$$= \frac{1}{T}\int_o^T \mathcal{E}\{Y_n(t)\}\ dt.$$

By the definition of h_n we find

$$\mathcal{E}\{Y_n(t)\} = \int_{a_t}^{\infty} [x-a(t)]^n\ \phi(x)\ dx$$

where $\phi(x)$ is the standardized normal density function.

Hence we have

$$\mathcal{E}\{Z_n\} = \frac{1}{T}\int_o^T\int_{a_t}^{\infty} [x-a(t)]^n\ \phi(x)\ dxdt\ . \tag{5.2.2}$$

Using the binomial expansion for $[x-a(t)]^n$ the integral of the form $\int_c^{\infty} (x-c)^n\ \phi(x)dx$ may be evaluated as a finite sum of incomplete gamma functions. This would give a useful form for computing purposes.

For the variance of Z_n we first note that

$$\mathrm{Var}\{Z_n\} = \mathcal{E}\{\frac{1}{T^2}\int_o^T\int_o^T Y_n(t)\ Y_n(s)dsdt\} - \frac{1}{T^2}\int_o^T\int_o^T \mathcal{E}\{Y_n(t)\}\ \mathcal{E}\{Y_n(s)\}dsdt \tag{5.2.3}$$

$$= \frac{1}{T^2}\int_o^T\int_o^T \mathrm{Cov}[Y_n(t),\ Y_n(s)]\ dsdt\ .$$

Now

$$\mathcal{E}\{Y_n(t)\ Y_n(s)\} = \int_{a_t}^{\infty}\int_{a_s}^{\infty} [x-a(t)]^n\ [y-a(s)]^n\ \phi(x,y;r)\ dxdy\ , \tag{5.2.4}$$

where $\phi(x,y;r)$ is the standardized bivariate normal density with correlation coefficient $r = r(t-s)$.

In evaluating the right-hand side of (5.2.4) the following expansion, which may be found in Cramér (1946, p. 290) will be useful.

$$\phi(x,y;r) = \sum_{j=1}^{\infty} \frac{r^{j-1}}{(j-1)!} \Phi^{(j)}(x)\, \Phi^{(j)}(y) \tag{5.2.5}$$

where $|r| < 1$ and $\Phi^{(j)}(x)$ denotes the j-th derivative of the normal distribution function $\Phi(x)$.

Substitution of this expansion into (5.2.4) and formal interchange of summation and integration (the justification will be given in Section 3) yields

$$\mathcal{E}\{Y_n(t)\ Y_n(s)\} = \sum_{j=1}^{\infty} \frac{r^{j-1}}{(j-1)!} \int_{a_t}^{\infty} (x-a_t)^n\, \Phi^{(j)}(x)dx \cdot \int_{a_s}^{\infty} (y-a_s)^n\, \Phi^{(j)}(y)dy \tag{5.2.6}$$

which holds at least for a.e.(t,s).

For j such that $1 \leq j \leq n+1$, repeated integration by parts gives

$$\int_a^{\infty} (x-a)^n\, \Phi^{(j)}(x)dx = (-1)^{j-1} \frac{n!}{(n-j+1)!} \int_a^{\infty} (x-a)^{n-j+1}\, \phi(x)dx ,$$

and if $j > n+1$

$$\int_a^{\infty} (x-a)^n\, \Phi^{(j)}(x)dx = (-1)^{n+1} n!\, \Phi^{(j-n-1)}(a) .$$

Hence (5.2.6) may be written as

$$\mathcal{E}\{Y_n(t)\ Y_n(s)\} = \sum_{j=0}^{n} \frac{r^j}{j!} \left[\frac{n!}{(n-j)!}\right]^2 \int_{a_t}^{\infty} (x-a_t)^{n-j}\, \phi(x)dx \cdot \int_{a_s}^{\infty} (y-a_s)^{n-j}\, \phi(y)\, dy$$

$$+ (n!)^2 \sum_{j=1}^{\infty} \frac{r^{n+j}}{(n+j)!} \Phi^{(j)}(a_t)\, \Phi^{(j)}(a_s) .$$

Note that the first term (j = 0) of the finite sum is

$$\int_{a_t}^{\infty} (x-a_t)^n\, \phi(x)dx \cdot \int_{a_s}^{\infty} (y-a_s)^n\, \phi(y)dy$$

which is just $\mathcal{E}\{Y_n(t)\} \cdot \mathcal{E}\{Y_n(s)\}$.

Thus

$$\mathrm{Cov}[Y_n(t),Y_n(s)] = \sum_{j=1}^{n} \frac{r^j}{j!} \left[\frac{n!}{(n-j)!}\right]^2 \int_{a_t}^{\infty} (x-a_t)^{n-j}\, \phi(x)dx \cdot \int_{a_s}^{\infty} (y-a_s)^{n-j}\, \phi(y)dy$$

$$+ (n!)^2 \sum_{j=1}^{\infty} \frac{r^{n+1}}{(n+j)!}\, \Phi^{(j)}(a_t)\, \Phi^{(j)}(a_s) .$$

Substituting this into (5.2.3) and again postponing the justification of the interchange of summation and integration we obtain the final result

$$\mathrm{Var}\{Z_n\} = (\tfrac{n!}{T})^2 \{ \sum_{j=1}^{n} \frac{1}{j![(n-j)!]^2} \int_o^T\int_o^T [r^j(t-s) \int_{a_t}^{\infty} (x-a_t)^{n-j}\phi(x)dx \int_{a_s}^{\infty} (y-a_s)^{n-j}\phi(y)dy]dsdt \tag{5.2.7}$$

$$+ \sum_{j=1}^{\infty} \frac{1}{(n+j)!} \int_o^T\int_o^T r^{n+j}(t-s)\, \Phi^{(j)}(a_t)\, \Phi^{(j)}(a_s)dsdt\}$$

where it is understood that the first summation does not appear if n = 0.

For the important special case when a(t) = a, a constant, formulae (5.2.2) and (5.2.7) reduce to

$$\mathcal{E}\{Z_n\} = \int_a^{\infty} (x-a)^n\, \phi(x)dx ,$$

(5.2.8)

$$\mathrm{Var}\{Z_n\} = 2\frac{(n!)^2}{T} \{ \sum_{j=1}^{n} \frac{[\int_a^{\infty} (x-a)^{n-j}\phi(x)dx]^2}{j!\ [(n-j)!]^2} \int_o^T (1-\tfrac{\tau}{T})r^j(\tau)d\tau$$

$$+ \sum_{j=1}^{\infty} \frac{[\Phi^{(j)}(a)]^2}{(n+j)!} \int_o^T (1-\tfrac{\tau}{T})r^{n+j}(\tau)d\tau\} ,$$

where use is made of the fact that for "any" even function $f(\tau)$

$$\int_o^T\int_o^T f(t-s)dsdt = 2\int_o^T (T-\tau)f(\tau)d\tau .$$

These results have been announced without proof in Leadbetter and Cryer (1965c).

As noted in the introduction to this chapter, for any fixed n we may obtain an upper bound to the probability that the process will never exceed a given level or curve a(t) in the time period [0,T]. Specifically we have

(5.2.9) $P\{X(t) \leq a(t),\ t \in [0,T]\} = P\{Z_n = 0\} \leq (1 + [\mathcal{E}\{Z_n\}]^2/\mathrm{Var}\{Z_n\})^{-1}$

using a one-sided Chebyshev inequality.

5.3 Convergence Questions.

To justify the interchange of summation and integration in both (5.2.6) and (5.2.7) we need to look at the properties of the functions $\Phi^{(j)}(x)$. It is well known that these derivatives are related to the Hermite polynomials $H_j(x)$,

(5.3.1) $$\phi(x)2^{-j/2} H_j(x/2^{\frac{1}{2}}) = (-1)^j \Phi^{(j+1)}(x)$$

and thus known properties of the Hermite polynomials can be used. From Erdélyi (1953) we have

(5.3.2) $$e^{-\frac{1}{2}x^2} |H_j(x)| \leq k2^{j/2}(j!)^{\frac{1}{2}}$$

where k is an (absolute) constant.

Thus

(5.3.3) $$|\Phi^{(j)}(x)| \leq K[(j-1)!]^{\frac{1}{2}} e^{-\frac{1}{4}x^2}$$

where K is a constant.

To justify (5.2.6) it is sufficient to show that the series

(5.3.4) $$\sum_{j=1}^{\infty} \frac{|r|^{j-1}}{(j-1)!} \int_{a_t}^{\infty} (x-a_t)^n |\Phi^{(j)}(x)| dx \cdot \int_{a_s}^{\infty} (y-a_s)^n |\Phi^{(j)}(y)| dy$$

is convergent.

By (5.3.3) we have that

$$\int_{a_t}^{\infty} (x-a_t)^n |\Phi^{(j)}(x)| dx \leq K[(j-1)!]^{\frac{1}{2}} \int_{a_t}^{\infty} (x-a_t)^n e^{-\frac{1}{4}x^2} dx$$

$$\leq K_1[(j-1)!]^{\frac{1}{2}}$$

Thus the j-th term of (5.3.4) is less than $K_1^2|r(t-s)|^{j-1}$ and thus (5.3.4) converges at least for almost every $(s,t) \in [0,T] \times [0,T]$. This is all we need since we next want to integrate over s and t.

To justify (5.2.7) it is sufficient to establish the convergence of

$$\sum_{j=1}^{\infty} \frac{1}{(j+n)!} \int_0^T \int_0^T |r(t-s)|^{n+j} |\Phi^{(j)}(a_t)\Phi^{(j)}(a_s)| dsdt . \tag{5.3.5}$$

From the theory of characteristic functions (for example) we have

$$\Phi^{(j)}(x) = \frac{(-1)^{j-1}}{2\pi} \frac{d^{j-1}}{dx^{j-1}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}t^2-itx} dt$$

$$= \frac{(-i)^{j-1}}{2\pi} \int_{-\infty}^{\infty} t^{j-1} e^{-\frac{1}{2}t^2-itx} dt .$$

Therefore

$$|\Phi^{(j)}(x)| \le \frac{1}{\pi} \int_0^{\infty} t^{j-1} e^{-\frac{1}{2}t^2} dt$$

$$= \frac{2^{\frac{1}{2}j-1}}{\pi} \Gamma(\tfrac{1}{2}j)$$

Using Stirling's formula again we find that

$$\frac{|\Phi^{(j)}(a_t)\Phi^{(j)}(a_s)|}{(j+n)!} \le K/j^{n+3/2}$$

where K is a constant.

Further $\int_0^T \int_0^T |r(t-s)|^{n+j} dsdt \le T^2$.

Thus for any fixed n = 0,1,2,..., the series (5.3.5) converges as required.

5.4 Asymptotic Formulae and Numerical Computations.

In developing some limiting results as $T \to \infty$ we will restrict attention to the special case where $a(t) \equiv a$, a constant. Hence we are concerned with Equations (5.2.8). We note in particular that $\mathcal{E}\{Z_n\}$ does not depend on T.

Suppose that the covariance function $r(\tau)$ tends to zero as $|\tau| \to \infty$ and is in fact (absolutely) integrable on $(0,\infty)$. Then $r^j(\tau)$ for any positive integer j is also integrable and by dominated convergence

$$(5.4.1)\quad \int_0^T (1-\tfrac{\tau}{T}) r^j(\tau)d\tau \longrightarrow \int_0^\infty r^j(\tau)d\tau = a_j, \text{ say, as } T \longrightarrow \infty .$$

Hence using dominated convergence together with the convergence results of Section 3 we have

$$(5.4.2)\quad T\,\mathrm{Var}\{Z_n\} \longrightarrow 2(n!)^2\{\sum_{j=1}^{n} \frac{[\int_a^\infty (x-a)^{n-j}\phi(x)dx]^2}{j![(n-j)]^2} a_j + \sum_{j=1}^{\infty} \frac{[\Phi^{(j)}(a)]^2}{(n+j)!} a_{n+j}\}.$$

For a stationary normal Markov process, i.e. $r(\tau) = e^{-\alpha|\tau|}$ for some positive α, we obtain $a_j = (\alpha j)^{-1}$ so that (5.4.2) becomes for n = 1

$$(5.4.3)\qquad T\,\mathrm{Var}\{Z_1\} \longrightarrow \frac{2}{\alpha}\{[1-\Phi(a)]^2 + \sum_{j=1}^{\infty} \frac{[\Phi^{(j)}(a)]^2}{(j+1)(j+1)!}\} .$$

For non-integrable covariance functions the results can be very simple indeed. For example if,

$$r(\tau) \sim A/|\tau| ,$$

then

$$\int_0^T (1-\tfrac{\tau}{T})r(\tau)d\tau = (\int_0^1 + \int_1^T)(1-\tfrac{\tau}{T})r(\tau)d\tau$$

$$\sim \int_0^1 (1-\tfrac{\tau}{T})r(\tau)d\tau + A \log T$$

$$\sim A \log T \quad , \quad \text{as } T \longrightarrow \infty.$$

Further for $m > 1$, $\int_0^T (1-\frac{\tau}{T})r^m(\tau)d\tau$ converges to a finite limit as $T \longrightarrow \infty$ and so the first term of the variance formula of (5.2.8) dominates and

$$(5.4.4)\qquad \frac{T}{\log T}\,\mathrm{Var}\{Z_n\} \longrightarrow 2nA[\int_a^\infty (x-a)^{n-1}\phi(x)dx]^2$$

To obtain numerical results for the various intergals and infinite series occuring in this chapter the following remarks can be useful.

Integrals of the form $\int_a^\infty (x-a)^m \phi(x)dx$ where m is an integer appear both in $\mathcal{E}\{Z_n\}$ and in the finite sum involved in $\mathrm{Var}\{Z_n\}$. As noted previously these may evaluated using the binomial theorem and incomplete gamma functions, viz.

$$(5.4.5) \qquad \int_a^\infty (x-a)^m \phi(x)dx = \pi^{-\frac{1}{2}} \sum_{i=0}^{m} \binom{m}{i}(-a)^{m-i}\, 2^{\frac{1}{2}i-1}\, \Gamma\left(\frac{i+1}{2}, \frac{a^2}{2}\right)$$

where $\Gamma(m,x) = \int_x^\infty t^{m-1} e^{-t}\, dt$ is an incomplete gamma function. These functions are extensively tabled in Pearson (1957). The recurrence relation

$$(5.4.6) \qquad \Gamma(m+1,x) = m\Gamma(m,x) + x^m e^{-x}$$

may also be of use. Unfortunately, however, the first argument of Γ in (5.4.5) increases by half integers so that two such recurrence equations and two initial values are necessary to obtain all the terms in the sum. The initial values may easily be found using only standard normal distribution tables since

$$\Gamma(\tfrac{1}{2},\tfrac{1}{2}x^2) = 2\pi^{\frac{1}{2}}[1-\Phi(x)]$$

and

$$\Gamma(1,\tfrac{1}{2}x^2) = (2\pi)^{\frac{1}{2}}\phi(x) \;.$$

In approximating series of the form

$$\sum_{j=1}^{\infty} c_j [\Phi^{(j)}(a)]^2$$

it is useful to note

$$(5.4.7) \qquad \Phi^{(m+2)}(x) + x\Phi^{(m+1)}(x) + m\Phi^{(m)}(x) = 0 \qquad m \geq 1$$

which may be derived by partial integration or from known results for Hermite polynomials.

As an example, for the Markov case, $r(\tau) = e^{-\alpha|\tau|}$, and particular values of n, a, and αT (α is only a scale parameter on the time axis) the following numerical results were obtained.

Table 5.4.1 Numerical Results

$$a = 1, \ \alpha T = 2, \ r(\tau) = e^{-\alpha|\tau|}$$

n	$\mathcal{E}\{Z_n\}$	$\mathrm{Var}\{Z_n\}$	Chebyshev bound
0	.1587	.04721	.65
1	.0833	.02850	.80
2	.0641	.04799	.92
3	.0912	.13270	.94

The best bound is obtained for $n = 0$ and we have

$$P\{X(t) < 1, \ 0 \leq t \leq 2/\alpha\} \ \leq \ .65 \quad .$$

CHAPTER VI

TWO-SIDED BARRIERS

6.1 Introduction.

As noted in Chapter I most of the results of Chapters II - V can be extended to the case of two-sided barriers; that is, where we are interested in excursions outside both a (positive) level a and a (negative) level b. The mean of the total number of crossings of the level a and the level b is of course trivial to obtain; the results will be stated in Section 2. The derivation of the variance or second moment of the number of such crossings reduces to the calculation of the covariance between the number of crossings of the level a with the number of crossings of the level b. The methods of Chapter IV may be adapted to obtain this and a sketch of the proof is given in Section 3. A heuristic derivation of this covariance was given previously by Miller and Freund (1956). Finally the extension of the results on Z_n exceedance measures (as given in Chapter V) is discussed in Section 4. Two extensions are considered. (cf. Leadbetter (1963)).

Let N^a and N^b denote, respectively, the number of crossings of the level a and the level b for $0 \leq t \leq 1$ and let $N^{a,b}$ be the number of crossings of either level, i.e., $N^{a,b} = N^a + N^b$. Then $N^{a,b}$ is a two-sided barrier version of the number of crossings of a single level. In some contexts one might want to consider, say, the number of upcrossings of the level a plus the number of downcrossings of the level b. Further we could obviously consider crossings of two curves a(t), b(t). To conserve notation we will give explicit results only for certain cases. The corresponding formulae for other situations will be similar.

6.2 Mean Number of Crossings.

Suppose X(t) is a non-stationary normal process with mean $\mathcal{E}\{X(t)\} = m(t)$ and covariance $\mathrm{Cov}[X(t),X(s)] = \Gamma(t,s)$. Then with the notation of Theorem 3.3.1 we have immediately

Theorem 6.2.1: If the conditions of Theorem 3.3.1 hold, then

$$\mathcal{E}\{N^{a,b}\} = \mathcal{E}\{N^a\} + \mathcal{E}\{N^b\} \tag{6.2.1}$$

where $\mathcal{E}\{N^a\}$ and $\mathcal{E}\{N^b\}$ are given by (3.3.1) with $m(t)$ replaced by $m(t)-a$ and by $m(t)-b$, respectively.

Corollary 6.2.1: If $X(t)$ is stationary with mean zero and covariance $r(\tau)$, then

$$\mathcal{E}\{N^{a,-a}\} = \frac{2}{\pi}[-r''(0)/r(0)]^{\frac{1}{2}} \, e^{-a^2/[2r(0)]} . \tag{6.2.2}$$

6.3 The Variance.

To obtain the variance or mean square of $N^{a,b} = N^a + N^b$ clearly we need only derive a formula for $\mathcal{E}\{N^a N^b\}$, i.e., the (uncorrected) covariance between the number of crossings of the level a and the number of crossings of the level b. In order to have a direct analogy with the derivations of Chapter IV we consider only upcrossings and write $N_u^{a,b} = N_u^a + N_u^b$ in an obvious notation. The proof of the following main result will be given by means of several lemmas.

Theorem 6.3.1: Under the conditions of Theorem 4.2.1 we have the finite moment

$$\mathcal{E}\{N_u^a \, N_u^b\} = \int_0^1\int_0^1\int_0^\infty\int_0^\infty xy \, p_{t-s}(a,b,x,y)dxdy \, dsdt, \quad (a \neq b) , \tag{6.3.1}$$

where $p_\tau(u,v,x,y)$ is the joint density of $X(0)$, $X(\tau)$, $X'(0)$, $X'(\tau)$ as before.

Throughout the following lemmas N_n^a and N_n^b will denote the number of upcrossings of the levels a and b, respectively, by the linear process $Y_n(t)$ defined in Chapters II and III. We use notation from Chapter IV without further comment.

Lemma 6.3.1: For $a \neq b$ we have

$$N_n^a \, N_n^b = P + Q + \lim_{m\to\infty} \int\int_{S_n} \delta_m[Y_n(t)-a] \, \delta_m[Y_n(s)-b] \, \sigma[Y_n'(t)] \, \sigma[Y_n'(s)]dsdt \tag{6.3.2}$$

and

$$(6.3.3)\qquad \int\int_{S_n} \delta_m[Y_n(t)-a]\ \delta_m[Y_n(s)-b]\ \sigma[Y_n'(t)]\ \sigma[Y_n'(s)] \le 2^{2n}\ ,$$

where P denotes the number of intervals $(k2^{-n},\ (k+1)2^{-n})$ of (0,1) which contain both an upcrossing of a and an upcrossing of b by $Y_n(t)$ and Q denotes the numbers of such subintervals which contain an upcrossing of a and such that there is an upcrossing of b in an adjacent such interval.

Proof: Write $\alpha_k = k2^{-n}$. Then as in the proof of Lemmas 4.2.1 and 4.2.2 we have

$$N_n^a N_n^b = \lim_{m\to\infty} \sum_{k=0}^{2^n-1} \int_{\alpha_k}^{\alpha_{k+1}} \delta_m[Y_n(t)-a]\ \sigma[Y_n'(t)]dt \int_{\alpha_k}^{\alpha_{k+1}} \delta_m[Y_n(s)-b]\ \sigma[Y_n'(s)]ds$$

$$+ \lim_{m\to\infty} \sum_{k=0}^{2^n-2} \left[\int_{\alpha_k}^{\alpha_{k+1}} \int_{\alpha_{k+1}}^{\alpha_{k+2}} + \int_{\alpha_{k+1}}^{\alpha_{k+2}} \int_{\alpha_k}^{\alpha_{k+1}}\right] \delta_m[Y_n(t)-a]\ \delta_m[Y_n(s)-b]$$

$$\sigma[Y_n'(t)]\ \sigma[Y_n'(s)]dsdt$$

$$+ \lim_{m\to\infty} \int\int_{S_n} \delta_m[Y_n(t)-a]\ \delta_m[Y_n(s)-b]\ \sigma[Y_n'(t)]\ \sigma[Y_n'(s)]dsdt\ .$$

As in previous work we see that the first term on the right-hand side is just P. Similarly the second term is Q. The inequality (6.3.3) is analogous to that of Lemma 4.2.2.

To show that $\mathcal{E}\{P\}$ tends to zero as $n \to \infty$ we appeal to the theory of "streams of events" or "point processes" as given, for example, in the book of A. Y. Khintchine (1960). In our particular case we say that an event takes place at time τ if X(t) has either an upcrossing of the level a or an upcrossing of the level b at time τ. A stream is called stationary if for any set of non-negative integers $m_1, m_2, \ldots, m_k$ and any set of disjoint intervals (τ_i, τ_i'), $i = 1,\ldots,k$ the joint probability of m_i events in the interval $(\tau_i+T,\ \tau_i'+T)$ is the same for every T. For a stationary stream let $\omega(\tau)$ be the probability of at least two events in an interval of length τ. Then the stream is called orderly (or regular) if $\omega(\tau) = o(\tau)$ as $\tau \longrightarrow 0$. Sufficient conditions for a stationary stream to be orderly are that the mean number of events per unit time be finite and that there be zero probability of the simultaneous occurrence of two events anywhere in the interval (0,1). As pointed out by Leadbetter (1966b), this follows from an extension of a lemma of Dobrushin given by Volkonski (1960).

For the particular stream under consideration, the simultaneous occurrence of two events is impossible (with probability one) since $a \neq b$ and the sample functions are continuous (recall that from our definition, points of inflection are counted as only one crossing). Further since $\lambda_2 < \infty$ the mean number of events per unit time is finite. To show that the stream is stationary we write $N(\tau_i, \tau_i')$ for the number of upcrossings of a or b in (τ_i, τ_i'). By dividing each interval (τ_i, τ_i') into $2^n + 1$ equal parts we may define a linear process $Y_n(t)$ on each and thus obtain $N_n(\tau_i, \tau_i')$ the number of upcrossings of a or b by $Y_n(t)$ in (τ_i, τ_i'). From Lemma 2.2.2 we have that, with probability one, $N_n(\tau_i, \tau_i') \longrightarrow N(\tau_i, \tau_i')$ as $n \longrightarrow \infty$ $(i=1,2,\ldots k)$. Thus the joint distribution converges, i.e. we have

$$P\{N(\tau_i + T,\ \tau_i + T) = m_i,\ i=1,\ldots k\} \quad = \quad \lim_{n \to \infty} \ P\{N_n(\tau_i + T,\ \tau_i' + T) = m_i,\ i=1,\ldots k\} \ .$$

Now the probability on the right-hand side can be written as a certain integral of a finite-dimentional normal density. The range of integration does not depend on T and, since the process X(t) is stationary, the density does not depend on T. Hence the left-hand side does not depend on T and the stream of events is stationary.

Thus we may state that the probability of two or more events in an interval of length 2^{-n} is $o(2^{-n})$ as $n \longrightarrow \infty$.

Now consider the original problem. Assume with no loss of generality that $b < a$. Then

$$\mathcal{E}\{P\} = \sum_{k=0}^{2^n-1} P\{X(k2^{-n}) < b < a < X((k+1)2^{-n})\}$$

$$\leq \sum_{k=0}^{2^n-1} P\{\text{two events occur in } (k2^{-n},\ (k+1)2^{-n})\}$$

$$= 2^n\ P\{\text{two events occur in } (k2^{-n},\ (k+1)2^{-n})\}$$

$$= o(1)\ , \quad \text{as } n \longrightarrow \infty\ .$$

Therefore

$$\mathcal{E}\{P\} = o(1)\ , \quad \text{as } n \longrightarrow \infty\ .$$

Further

$$\mathcal{E}\{Q\} = \sum_{k=0}^{2^n-2} P\{X(k2^{-n}) < b < X((k+1)2^{-n}) < a < X((k+2)2^{-n})\}$$

$$\leq \sum_{k=0}^{2^n-2} P\{X(k2^{-n}) < b < a < X((k+2)2^{-n})\}$$

Hence

$$\mathcal{E}\{Q\} = o(1)\ , \quad \text{also.}$$

Thus using inequality (6.3.3), dominated convergence and Fubini's theorem we have

<u>Lemma</u> 6.3.2: As $n \longrightarrow \infty$

$$\mathcal{E}\{N_n^a N_n^b\} = \lim_{m\to\infty} \int\int_{S_n} \mathcal{E}\{\delta_m[Y_n(t)-a]\ \delta_m[Y_n(s)-b]\ \sigma[Y_n'(t)]\ \sigma[Y_n'(s)]\}\,ds\,dt$$

$$+\ o(1)\ .$$

Proof of the Theorem: Exactly as in Lemma 4.2.6 we now find

$$\lim_{m\to\infty} \iint_{S_n} \mathcal{E}\{\delta_m[Y_n(t)-a]\, \delta_m[Y_n(s)-b]\, \sigma[Y_n'(t)]\, \sigma[Y_n'(s)]\}dsdt$$

(6.3.6)

$$= \iint_{S_n} \int_0^\infty \int_0^\infty xy\, p_{n,t,s}(a,b,x,y)dxdydsdt\ ,$$

where $p_{n,t,s}(u,v,x,y)$ is the joint normal density for $Y_n(t)$, $Y_n(s)$, $Y_n'(t)$, $Y_n'(s)$ as usual.

To show that we may take the limit as $n \to \infty$ under the integrations we proceed as in Lemmas 4.2.7 - 10 (we need only replace (a,a) by (a,b), essentially). In this way we find

$$\lim_{n\to\infty} \iint_{S_n} \int_0^\infty \int_0^\infty xy\, p_{n,t,s}(a,b,x,y)dxdydsdt$$

$$= \int_0^1\int_0^1 \int_0^\infty \int_0^\infty xy\, p_{t-s}(a,b,x,y)dxdydsdt$$

and retracing steps (6.3.6), (6.3.5), and (6.3.4) completes the proof.

6.4 Exceedance Measures for Two-sided Barriers.

In Chapter V we considered certain random variables Z_n which in a sense measured the extent to which the process X(t) exceeded the barrier a(t). In many contexts it is perhaps more realistic to consider exceedances both above a(t) and below b(t), say - that is, a two-sided barrier.

We assume, as in Chapter V, that X(t) is a stationary normal process with mean zero, covariance function $r(\tau)$, normalized so that $r(0) = 1$. We further assume that $|r(\tau)| < 1$ a.e. and that, with probability one, X(t) has everywhere continuous sample functions.

To simplify many of the formulae we further restrict attention to the case where $a(t) \equiv a$, a constant, and $b(t) \equiv -a$. The more general situation can be handled in the same manner but the formulae do not simplify.

For each non-negative integer n we now define

$$g_n(x) = \begin{cases} (x-a)^n & \text{if } x \geq a \\ (-a-x)^n & \text{if } x \leq -a \\ 0 & \text{otherwise} \end{cases}$$

and then let

$$W_n = \frac{1}{T} \int_o^T g_n[X(t)]dt \ . \tag{6.4.1}$$

The quantity W_o is the proportion of time for which the process is either above a or below -a and TW_1 is the total area which X(t) cuts off above a plus the area below -a.

From symmetry considerations we immediately have

$$\begin{aligned} \mathcal{E}\{W_n\} &= 2\ \mathcal{E}\{Z_n\} \\ &= 2 \int_a^\infty (x-a)^n\ \phi(x)dx \end{aligned} \tag{6.4.2}$$

To obtain the variance of W_n we need now consider $\mathrm{Cov}\{g_n[X(t)],\ g_n[X(s)]\}$. We have

$$\mathcal{E}\{g_n[X(t)]\ g_n[X(s)]\}$$

$$= [\int_a^\infty \int_a^\infty (x-a)^n(y-a)^n + 2 \int_{-\infty}^{-a} \int_a^\infty (x-a)^n(-a-y)^n + \int_{-\infty}^{-a} \int_{-\infty}^{-a} (-a-x)^n(-a-y)^n] \ \phi(x,y;r)dxdy$$

$$= 2[\int_a^\infty \int_a^\infty (x-a)^n(y-a)^n\ \phi(x,y;r)dxdy + \int_a^\infty \int_a^\infty (x-a)^n(y-a)^n\ \phi(x,y;-r)dxdy]$$

where, as previously, $\phi(x,y;r)$ is the standardized bivariate normal density with correlation coefficient r. In our case $r = r(t-s)$.

Hence we can use the expansion (5.2.5) again but this time half of the resulting terms will cancel and we are led to the following result.

If n is even (the first sum does not appear if n = 0)

$$\begin{aligned}\mathrm{Var}\{W_n\} &= 8\frac{(n!)^2}{T}\{\sum_{j=1}^{n/2}\frac{[\int_a^{\infty}(x-a)^{n-2j}\phi(x)dx]^2}{(2j)![(n-2j)!]^2}\cdot\int_o^T(1-\tfrac{\tau}{T})r^{2j}d\tau \\ &\quad + \sum_{j=1}^{\infty}\frac{[\phi^{(2j)}(a)]^2}{(n+2j)!}\int_o^T(1-\tfrac{\tau}{T})r^{n+2j}d\tau\}\end{aligned} \tag{6.4.3}$$

and if n is odd (the first sum does not appear if n = 1)

$$\begin{aligned}\mathrm{Var}\{W_n\} &= 8\frac{(n!)^2}{T}\{\sum_{j=1}^{(n-1)/2}\frac{[\int_a^{\infty}(x-a)^{n-2j}\phi(x)dx]^2}{(2j)![(n-2j)!]^2}\int_o^T(1-\tfrac{\tau}{T})r^{2j}d\tau \\ &\quad + \sum_{j=1}^{\infty}\frac{[\phi^{(2j-1)}(a)]^2}{(n+2j-1)!}\int_o^T(1-\tfrac{\tau}{T})r^{n+2j-1}d\tau\}\end{aligned} \tag{6.4.4}$$

In defining $g_n(x)$ as we have we are treating positive and negative excursions outside the bounds a,-a as being equally "bad" from a performance point-of-view. However in other situations it may be that negative excursions can in fact compensate for positive excursions. In such a case we can define $g_n^*(x)$, say, as

$$g_n^*(x) = \begin{cases}(x-a)^n & \text{if } x \geq a \ , \\ -(-a-x)^n & \text{if } x \leq -a \ , \\ 0 & \text{otherwise} \ .\end{cases}$$

We then let

$$W_n^* = \frac{1}{T}\int_o^T g_n^*[X(t)]dt \ .$$

The quantities W_o^* and W_1^* may be easily interpreted. W_o^* represents the difference between the proportion of time during (0,T) for which $X(t) \geq a$ and the proportion of time for which $X(t) \leq -a$. Similarly TW_1^* is the difference between the area which the process cuts off above a and the corresponding area below -a.

By the definition of $g_n^*(x)$ we clearly have

$$\mathcal{E}\{W_n^*\} = 0 , \tag{6.4.5}$$

and proceeding exactly as for $g_n(x)$ we obtain the following expression for the variance.

If n is even (the first sum does not appear if n = 0)

$$\begin{aligned} \operatorname{Var}\{W_n^*\} &= 8\frac{(n!)^2}{T}\Big\{\sum_{j=1}^{n/2} \frac{[\int_a^\infty (x-a)^{n-2j+1}\phi(x)dx]^2}{(2j-1)![(n-2j+1)!]^2} \int_o^T (1-\tfrac{\tau}{T})r^{2j-1}d\tau \\ &\quad + \sum_{j=1}^{\infty} \frac{[\Phi^{(2j-1)}(a)]^2}{(n+2j-1)!} \int_o^T (1-\tfrac{\tau}{T})r^{n+2j-1}d\tau)\Big\} \end{aligned} \tag{6.4.6}$$

and if n is odd

$$\begin{aligned} \operatorname{Var}\{W_n^*\} &= 8\frac{(n!)^2}{T}\Big\{\sum_{j=1}^{(n+1)/2} \frac{[\int_a^\infty (x-a)^{n-2j+1}\phi(x)dx]^2}{(2j-1)![(n-2j+1)!]^2} \int_o^T (1-\tfrac{\tau}{T})r^{2j-1}d\tau \\ &\quad + \sum_{j=1}^{\infty} \frac{[\Phi^{(2j)}(a)]^2}{(n+2j)!} \int_o^T (1-\tfrac{\tau}{T})r^{n+2j}d\tau\Big\} \end{aligned} \tag{6.4.7}$$

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APPENDIX

Lemma A.1: Let $X(t)$ be a stationary normal process with quadratic mean derivative $X'(t)$ existing and with spectral distribution function $F(\lambda)$, i.e.,

$$\text{cov}[X(t),X(t+\tau)] = \int_o^\infty \cos\lambda\tau \, dF(\lambda) .$$

Let $t_1,t_2,\ldots,t_k$ be distinct time points. Then if $F(\lambda)$ has a continuous component the joint distribution of $\underline{X} = (X(t_1), X(t_2),\ldots, X(t_n), X'(t_1), X'(t_2),\ldots, X'(t_n))$ is non-singular.

Proof: See Cramér and Leadbetter (1965).

Lemma A.2: If $\underline{x} = (x_1,\ldots,x_n)$, $\underline{y} = (y_{n+1},\ldots,y_m)$ are real row vectors and A is an m x m symmetric positive definite matrix, then

$$\min_{\underline{x}}[(\underline{x},\underline{y})A^{-1}(\underline{x},\underline{y})'] = \underline{y}A_3^{-1}\underline{y}' ,$$

where A_3 is the m-n x m-n matrix obtained by partitioning $A = \begin{pmatrix} A_1 & A_2 \\ A_2' & A_3 \end{pmatrix}$ corresponding to the dimensions of $\underline{x}$ and $\underline{y}$.

Proof: Let A^{-1} be partitioned similarly as $\begin{pmatrix} P_1 & P_2 \\ P_2' & P_3 \end{pmatrix}$. Then

$$\frac{d}{d\underline{x}}[(\underline{x},\underline{y})A^{-1}(\underline{x},\underline{y})'] = 2P_1\underline{x}' + 2P_2\underline{y}' .$$

Setting this equal to the zero vector and solving for $\underline{x}'$ yields

$$\underline{x}' = -P_1^{-1}P_2\underline{y}' .$$

We note that $\frac{d^2}{d\underline{x}d\underline{x}'}[(\underline{x},\underline{y})A^{-1}(\underline{x},\underline{y})'] = 2P_2$ which is positive definite and thus ensures that we have found a minimum.

The minimum value is

$$\underline{y}(P_3 - P_2' P_1^{-1} P_2)\underline{y}'$$

which gives the final result since $P_3 - P_2' P_1^{-1} P_2 = A_3^{-1}$; see Anderson (1958), for example.

Lemma A.3: For $|\rho| < 1$

$$\int_0^\infty \int_0^\infty xy \exp[-\tfrac{1}{2}(x^2 + 2\rho xy + y^2)]dxdy = (1-\rho^2)^{-1}[1 - \Delta\cot^{-1}\Delta] ,$$

where

$$\Delta = \rho(1-\rho^2)^{-\frac{1}{2}} \quad \text{and} \quad 0 \leq \cot^{-1}\Delta \leq \pi .$$

Proof: Let $I(\theta) = \int_0^\infty \int_0^\infty \exp[-\tfrac{1}{2}(x^2 + 2\cos\theta xy + y^2)]dxdy$.

If we make the transformation $x = u+v$, $y = u-v$ and then change to polar coordinates we find

$$I(\theta) = \theta \csc \theta .$$

Differentiating both sides with respect to θ yields

$$\sin\theta \int_0^\infty \int_0^\infty xy \exp[-\tfrac{1}{2}(x^2 + 2\cos\theta xy + y^2)]dxdy = \csc\theta(1 - \theta\cot\theta)$$

or

$$\int_0^\infty \int_0^\infty xy \exp[-\tfrac{1}{2}(x^2 + 2\cos\theta xy + y^2)]dxdy = \csc^2\theta(1 - \theta\cot\theta) .$$

Now put $\cos\theta = \rho$. Using (for principal values)

$$\cos^{-1}\rho = \csc^{-1}[(1-\rho^2)^{-\frac{1}{2}}] = \cot^{-1}[\rho(1-\rho^2)^{-\frac{1}{2}}]$$

we obtain the desired result.